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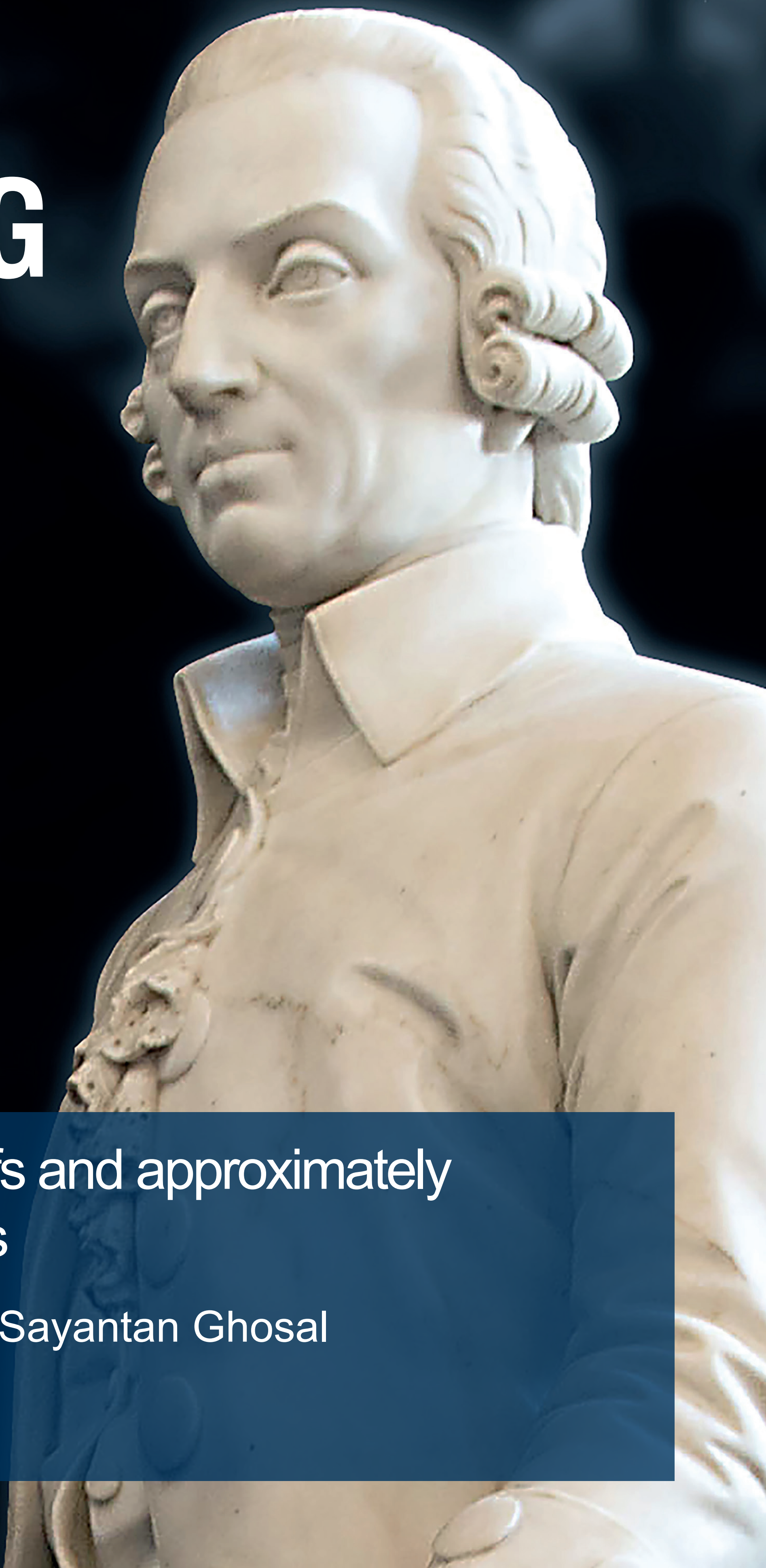

Heterogeneous beliefs and approximately self-fulfilling outcomes

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Paper no. 2021-07
May 2021

Heterogeneous beliefs and approximately self-fulfilling outcomes*

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May 14, 2021

Abstract

When are heterogenous beliefs compatible with equilibrium and if not, which non-equilibrium outcomes do they lead to? In this paper, we examine the conditions under which heterogenous beliefs lead to approximately self-fulfilling outcomes consistent with all that is commonly known by each agent via an iterative elimination process. We develop a formal definition of approximately self-fulfilling outcomes, $p-$consensus, and an associated, continuous measure of the degree of stability of equilibrium, $p-$stability. Applying our concepts to intertemporal trade in a two period economy, we examine how heterogenous beliefs and heterogenous preferences interact to create to asset price bubbles.

Keywords: $p-$consensus, $p-$stability, equilibrium, rationalizability, heterogeneous, beliefs, preferences, games, markets.

JEL Classifications C70 and D84.

*We would like to thank seminar/conference audiences at Warwick, Cergy-Pontoise, Barcelona, Atlanta for their helpful comments. Both authors would like acknowledge funding from Universite Cergy-Pontoise and University of Glasgow; Ghosal is grateful for funding from the ESRC funded Rebuilding Macro Network. E-mail: gabriel.desgranges@u-cergy.fr, Sayantan.Ghosal@glasgow.ac.uk.

1 Introduction

When do heterogeneous beliefs lead to equilibrium and if not, what kind of outcomes do they lead to?

In this paper, our starting point is that agents beliefs about outcomes are described by $p-$beliefs, namely, beliefs which put probability at least p on a specific outcome; with probability $1-p$ beliefs are heterogenous. We define a $p-$consensus outcome as one which, given $p-$beliefs on it, is rationalizable. Heuristically, a $p-$consensus outcome is approximately self-fulfilling where p is a measure of the degree of approximation.

The only outcome consistent with $1-$beliefs is an equilibrium one.[1] The closer p is to one, the lower is the degree of heterogeneity of beliefs; the closer p is to zero, the more heterogeneous beliefs are. For some value of p is less than one, we say an equilibrium is $p-$stable if is the unique rationalizable outcome with $p-$beliefs on it. The interpretation is that p is a continuous index for stability, a measure of the robustness of equilibrium to heterogeneous beliefs, less coarse that the usual "stable/unstable" typology.

Which equilibria are $p-$stable? Which features of an equilibrium does $p-$stability relate to? What non-equilibrium outcomes are also $p-$consensus outcomes? What insights can we obtain when we apply these concepts to the study of markets about the conditions under which non-equilibrium can be sustained and how can these non-equilibrium outcomes be characterized?

For ease of exposition, we initially work in a simple setting of strategic complements where the identical best response of each agent is the same function of the expected average action. We obtain an explicit characterization of link between a non-equilibrium outcome and the value of p required to sustain it as a $p-$consensus outcome. We show that corner equilibria have a higher degree of stability (corresponding to a lower bound on the degree "p" of their $p-$stability). In a linear example with one interior and two corner Nash equilibria, we obtain closed form solutions for $p-$consensus outcomes and calculate the bounds on the $p-$stability of equilibrium outcomes. These results show that a Nash equilibrium is robust to some degree of heterogeneity of beliefs while sustaining a non-equilibrium outcome as a $p-$consensus outcome requires that beliefs be sufficiently heterogenous (hence, the upper bound on p).

The two concepts introduced by us bring added value to the analysis of large games and markets only when there are multiple rationalizable outcomes. Our results explore this point in greater detail in a general strategic setting due to MasColell (1984).

Our first main result demonstrates the link between the set of rationalizable outcomes and the set of $p-$consensus outcomes: any outcome in the interior of the set of rationalizable outcomes is also $p-$consensus outcome for some $p > 0$, thus characterizing the set of outcomes that are "approximately" self-fulfilling.

[1] This observation echoes, in our setting, the seminal characterization of the epistemic conditions underpinning a Nash equilibrium in Aumann and Brandenburger (1995). Since each player knows the choices of the others, and is rational, his choice must be optimal given theirs; so any outcome must be a Nash equilibrium outcome.

We say that a Nash equilibrium is inadmissible if the best-response map is "vertical at the equilibrium" i.e. a small change in the other players actions implies a infinitely large change in any one player's best response. We show that every admissible Nash equilibrium is $p-$stable for some $p < 1$. This result provides a precise characterization of Nash equilibria which are robust to some degree of heterogeneity of beliefs.

Under a mild continuity restriction, we show that a $p-$stable equilibrium is a locally isolated $p-$consensus distribution i.e. there is no other $p-$consensus distribution in its vicinity. Under an additional interiority condition, we also prove the converse statement: if an equilibrium is not $p-$stable, then there is always a $p-$consensus outcome in its vicinity.

Further, we show that in smooth settings, $p-$stability is related to the inverse of the slope of the best-response, thus confirming the intuition that the slope of the best-response matters for the strategic stability of equilibrium. Moreover, in settings with multiple equilibria, the $p-$stability of a Nash equilibrium is related to the inverse of the slope of the best-response and not to the size of it's basin of attraction.

We, then, apply our analysis to re-examine the foundations of intertemporal trade in a two period economy with $p-$beliefs over second period prices. We assume that individuals submit demand functions so that belief coordination in current spot market prices is never a problem: instead the individuals need to coordinate beliefs about future spot market prices when they trade in current spot markets. We derive conditions under which a $p-$consensus outcome differs from perfect foresight equilibria. These conditions require preference heterogeneity over second period consumption. We show, when individual preferences are additive in consumption over the two periods and they have identical, homothetic preferences over second period commodities, second-period spot market prices do not depend on redistribution of revenue within that period. In this case, any $p-$consensus outcome must be a perfect foresight equilibrium outcome.

In a local analysis, we show that when (a) second period spot market prices are sensitive to redistributions of revenue in second period markets or a redistribution of commodities in period 1 change second period spot market prices, *and* (b) there is lack of consensus over second period prices (so that beliefs over second period prices are heterogeneous to a sufficient degree), then an asset price bubble exists. Conversely, even with lack of consensus over future prices in a small enough neighborhood of a perfect foresight equilibria, an asset price bubble will not exist if these conditions do not hold.

We, then, conduct a global analysis in a simple example with an unique PFE at which there is no trade in asset markets. In a global analysis, we demonstrate how the interaction of heterogeneous beliefs and heterogeneous preferences implies the existence of $p-$consensus outcomes characterized by asset price bubbles. The distance of a PFE asset price a $p-$consensus asset price is entirely constrained by the heterogeneity of preferences.

The notion of $p-$consensus and $p-$stability builds on the seminal work on eductive stability (Guesnerie (1992), Evans and Guesnerie (1993), Guesnerie (2005), Evans, Guesnerie and McCullogh (2019)). The analysis developed here differs in that we propose a new solution concept that allows belief heterogeneity and our measure of stability is continuous one. Related solution concepts include those allowing for heterogeneous beliefs off the equilibrium path of play (e.g. the notion of self-confirming equilibrium, Fudenberg and Levine (1993)) and in the macro literature by Angeletos, Collard and Dellas (2018) (where heterogenous beliefs are treated as a model parameter).

The next section considers the simple setting with strategic complements to introduce and illustrate the workings of our concepts. The next section is devoted to the study of a general model and provides a number of basic results and illustrates the application of the two concepts using examples. In section 4, we study partial consensus in a two period economy. The last section concludes. The appendix contains the proofs of most of our results stated in the main body of the paper.

2 A game with strategic complements

Consider a game with a continuum $[0, 1]$ of agents with identical preferences and action set. The action set is a compact interval $I \subset R$. Denote $a_i \in I$ denote an action agent i and $\bar{a}$ an average action. The belief of agent i is a distribution of average actions (an element in $\Delta(I)$). Denote $E_i(\bar{a})$ the mean value of i's belief. The best-response of agent i to his belief is

$$a_i = \phi(E_i(\bar{a})),$$

where ϕ is a non-decreasing and C^1 map ($\phi' \geq 0$). This is a game with strategic complements with the feature that only the mean average action is payoff relevant for determining an individual agent's best-response.

Assume that there is a finite number $N > 1$ of Nash equilibria $a_1^* < a_2^* < ... < a_N^*$. Normalize the action set for the smallest and the largest Nash actions to be -1 and 1. With some abuse of notation, let $\overline{a}$ to denote the Dirac measure on $\overline{a}$, $\overline{a} \in [-1, 1]$. A well-known result (Milgrom and Roberts (1990)) is that the set of rationalizable outcomes is $[-1, 1]$. Denote $\|\phi'\| = \sup_{[-1,1]} \phi'(\bar{a})$. Nash multiplicity implies $\|\phi'\| > 1$. Let $p_0 \in (0, 1)$ be the unique value such that $(1 - p_0)\|\phi'\| = 1$.

For a fixed $\overline{a} \in [-1, 1]$ and $p \in [0, 1]$, a $p-$belief assigns a probability p to the average action $\overline{a}$ and a probability $1 - p$ to some other $\overline{a}' \in [-1, 1]$. We define a $p-$consensus outcome iteratively as follows. For a fixed $\overline{a} \in [-1, 1]$ and $p \in [0, 1]$, let

$$S^0_{\overline{a},p} = \{\overline{a}'' : \overline{a}'' = p\overline{a} + (1-p)\overline{a}', \, \overline{a}' \in [-1, 1]\}.$$

Consider the sequence of sets $S^n_{\overline{a},p}(\bar{a}) = \left[\phi(S^{n-1}_{\overline{a},p})\right] \cap S^{n-1}_{\overline{a},p}$ for $n \geq 1$. This sequence is decreasing and therefore, it converges to a set $S^\infty_{\overline{a},p}$. Then, $\overline{a}$ is a $p-$consensus outcome if $\overline{a} \in S^\infty_{\overline{a},p'}$ for all $p' \leq p$.

Straightforwardly from the definition of a $p-$consensus outcome, every rationalizable outcome is a $0-$consensus outcome. Evidently, every $p-$consensus outcome, some $p \in [0,1]$, is also a rationalizable outcome and every rationalizable outcome $\bar{a} \in [-1,1]$ is a $p-$consensus outcome for some $p \in [0,1]$. Hence, the interpretation is that p is the maximum degree of consensus compatible with sustaining $\bar{a} \in [-1,1]$ as a rationalizable outcome.

Any non Nash outcome $\bar{a} \in [-1,1]$, must lie between two Nash outcomes ($\bar{a} \in \left(a_n^*, a_{n+1}^*\right)$). The following proposition characterizes the degree of consensus compatible with a non-equilibrium outcome $\bar{a} \in [-1,1]$:

Proposition 1. For any non Nash outcome $\bar{a} \in [-1,1]$: (a) if $\phi(\bar{a}) < \bar{a}$ for $\bar{a} \in \left(a_n^*, a_{n+1}^*\right)$, then the value of p required to sustain $\bar{a}$ as a $p-$consensus increases in $\bar{a}$ for $\bar{a} \in \left(a_n^*, a_{n+1}^*\right)$; (b) if $\phi(\bar{a}) > \bar{a}$ for $\bar{a} \in \left(a_n^*, a_{n+1}^*\right)$, then the value of p required to sustain $\bar{a}$ as a $p-$consensus decreases in $\bar{a}$ for $\bar{a} \in \left(a_n^*, a_{n+1}^*\right)$. If $\phi'(a_1^*) < 1$, then the value of p for which $\bar{a} \in \left(a_{2n+1}^*, a_{2n+2}^*\right)$ is a $p-$consensus outcome increases in $\bar{a}$ and decreases in $\bar{a}$ for $\bar{a} \in \left(a_{2n}^*, a_{2n+1}^*\right)$. If $\phi'(a_1^*) > 1$, the value of p for which $\bar{a} \in \left(a_{2n+1}^*, a_{2n+2}^*\right)$ decreases in $\bar{a}$, and increases in $\bar{a} \in \left(a_{2n}^*, a_{2n+1}^*\right)$.

Proof. Consider a non Nash rationalizable outcome $\bar{a}$ such that $\phi(\bar{a}) < \bar{a}$. For $p' \in [0,1]$ and $x \in [-1,1]$, denote $BR_{\bar{a},p'}(x) = \phi(p'\bar{a} + (1-p')x)$ where $BR_{\bar{a},p'}(x)$ is non-decreasing and C^1. For any $p' \in [0,1]$, $BR_{\bar{a},p'}(-1) \geq -1$ and $BR_{\bar{a},p'}(\bar{a}) < \bar{a}$, which implies that $BR_{\bar{a},p'}$ admits a fixed point in $[-1,\bar{a})$. Note that the limit set $S_{\bar{a},p'}^{\infty}$ is an interval whose bounds are the lowest and highest fixed points of $BR_{\bar{a},p'}$. This implies that $\bar{a} \in S_{\bar{a},p'}^{\infty}$ iff $BR_{\bar{a},p'}$ admits a fixed point in $[\bar{a},1]$. By continuity, the degree p of consensus of $\bar{a}$ is such that $BR_{\bar{a},p}$ admits a fixed $\bar{a}_f$ in $[\bar{a},1]$ and $BR'_{\bar{a},p}(\bar{a}_f) = 1$ (the map $BR_{\bar{a},p}$ is tangent to the $45°$ line at $\bar{a}_f$). Consider now another non Nash rationalizable outcome $\bar{a}'$ such that $\phi(\bar{a}') < \bar{a}'$ and $\bar{a}' > \bar{a}$. Since ϕ is increasing, $BR_{\bar{a}',p}(\bar{a}_f) > \bar{a}_f$ and by continuity $BR_{\bar{a}',p}$ admits a fixed point in $[\bar{a}', \bar{a}_f]$ (and even another one in $[\bar{a}_f, 1]$). It follows that $\bar{a}' \in S_{\bar{a}',p'}^{\infty}$ and the degree $\bar{p}'$ of consensus of is strictly larger than $\bar{p}$. This proves part (a). The proof of part (b), when $\phi(\bar{a}) > \bar{a}$ for a non Nash rationalizable outcome $\bar{a}$ is analogous (the core of the argument relies now on the existence of a fixed point in $[-1,\bar{a}]$). : When $\phi'(a_1^*) < 1$, the values $\bar{a}$ in (a_1^*, a_2^*) are such that $\phi(\bar{a}) < \bar{a}$, the values $\bar{a}$ in (a_2^*, a_3^*) are such that $\phi(\bar{a}) < \bar{a}$ and so on. Applying part (a) repeatedly, the required result. When $\phi'(a_1^*) > 1$, the inequalities go the other way. Applying part (b) repeatedly, the required result.■

We define a Nash equilibrium $\bar{a}^*$ to be $p-$stable if $\bar{a}^* = S_{\bar{a}^*,p'}^{\infty}$ for all $p' > p$, $p < 1$. As already noted the set of $1-$consensus outcomes and Nash equilibrium outcomes must coincide.

Assume $N = 3$ so that there are 3 Nash equilibria: two "corner" equilibria $a_1^* = -1$ and $a_3^* = 1$ and an "interior" Nash equilibrium $a_2^* \in (-1,1)$. We characterize the $p-$stability of Nash equilibria for the case of three equilibria with the following corollary to the preceding proposition:

Corollary 1. Assume $N = 3$. If a_2^* is such that $\phi'(a_2^*) < 1$, then every Nash outcome is $0-$stable and every non Nash rationalizable outcome is $0-$consensus. If a_2^* is such that $\phi'(a_2^*) > 1$, then a_2^* is $p-$stable for $p_2 > 0$ and a_1^* and a_3^* have lower stability index, i.e. they are p_1-stable and p_3-stable respectively with

$p_1 < p_2$ and $p_3 < p_2$

Proof. Consider the case $\phi'(a_2^*) < 1$. We have $\phi(x) > x$ for $x \in (a_1^*, a_2^*)$ and $\phi(x) < x$ for $x \in (a_2^*, a_3^*)$. Therefore, for any $p > 0$, $BR_{a_2^* p}(x) > x$ for $x \in [a_1^*, a_2^*)$ and $BR_{a_2^* p}(x) < x$ for $x \in (a_2^*, a_3^*]$. This implies that $S^{\infty}_{a_2^*, p}$ reduces to a single value (that is a_2^*). This proves $0-$stability of a_2^*. Proposition 1 immediately proves the other part of the result. Consider the case $\phi'(a_2^*) > 1$. We have $\phi(x) < x$ for $x \in (a_1^*, a_2^*)$ and $\phi(x) > x$ for $x \in (a_2^*, a_3^*)$. Therefore, for $p > 0$ close to 0, $BR_{a_2^*, p}(-1) \geq -1$ and $BR_{a_2^*, p}(x) < x$ for some $x \in (a_1^*, a_2^*)$, which implies that $BR_{a_2^*, p}$ admits a fixed point in $[-1, x]$ and this fixed point is strictly smaller than a_2^*. A similar argument shows that $BR_{a_2^*, p}$ admits another fixed point strictly larger than a_2^*. Hence, $S^{\infty}_{a_2^*, p}$ does not reduce to a single value. This proves $\bar{p}-$stability of a_2^* for some $\bar{p} > 0$. Proposition i1 immediately proves the other part of the result.■

The corollary demonstrates that the interesting implication is that the two "corner" equilibria are more stable than the "interior" one.

As an example to illustrate the preceding results, consider the following step-wise linear example where $I = [-1, 1]$ and the best-response map is as follows:

$$a_i = \phi(\overline{a}) = \begin{cases} \beta\bar{a} \text{ for } \bar{a} \in \left[\frac{-1}{|\beta|}, \frac{1}{|\beta|}\right] \\ -1 \text{ for } \bar{a} < \frac{-1}{|\beta|} \\ 1 \text{ for } \bar{a} > \frac{1}{|\beta|} \end{cases}$$

and $\beta > 0$. Note that $\frac{\beta}{|\beta|}$ is either -1 or 1 depending on the sign of β.

When $\beta < 1$, the unique Nash equilibrium outcome is $\overline{a}^* = \{0\}$ When $\beta > 1$, there are three Nash equilibria: $\overline{a}^* \in \{-1, 0, 1\}$.

By computation, note that when $|\beta| < 1$, the unique rationalizable outcome is the (unique) Nash equilibrium outcome $\overline{a}^* = 0$ (the equilibrium outcome is $0-$stable). When $\beta > 1$, by computation, it follows that (i) $\overline{a}^* = 0$ is $p-$stable for $p = 1 - \frac{1}{\beta}$, (ii) the equilibrium outcomes $\overline{a}^* = -1$ and $\overline{a}^* = 1$ are $p-$stable for $p = \frac{1}{2}\left(1 - \frac{1}{\beta}\right)$. If $\beta < 1$, no outcome different from a Nash equilibrium is a $p-$consensus outcome, and we restrict attention to the case $\beta > 1$ in what follows.

Corollary 2. For $\beta > 1$, every $\overline{a}$ that differ from any Nash equilibrium ($\overline{a} \notin \{-1, 0, 1\}$) is a $p-$consensus outcome for

$$p = \frac{1}{1 + |\bar{a}|}\left(1 - \frac{1}{\beta}\right) \in \left[\frac{1}{2}\left(1 - \frac{1}{\beta}\right), 1 - \frac{1}{\beta}\right].$$

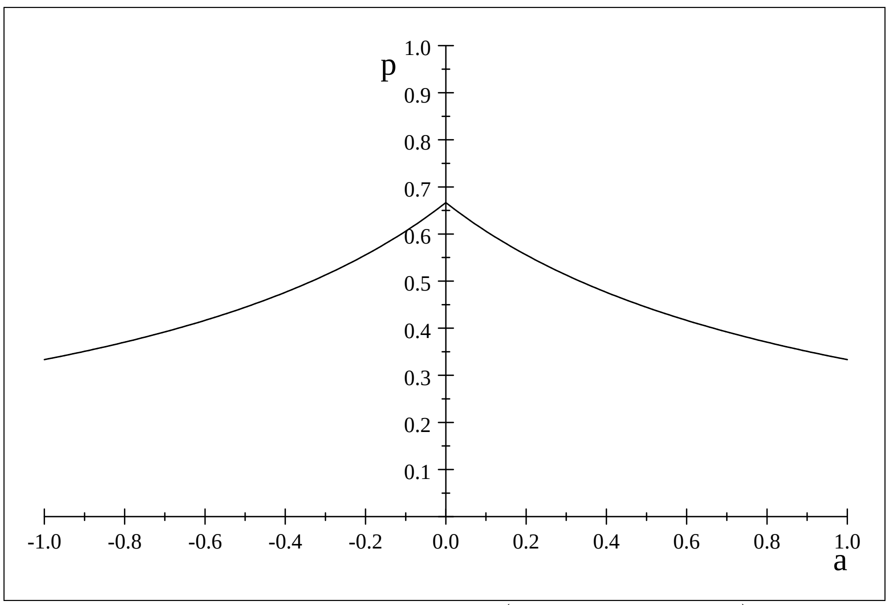
Proof. By computation, $BR_{\bar{a},p}(x) = \beta(p\bar{a} + (1-p)x)$ if $\beta(p\bar{a} + (1-p)x) \in [-1, 1]$ and $BR_{\bar{a},p}(x) = -1$ or 1 if $\beta(p\bar{a} + (1-p)x)$ is smaller than -1 or larger than 1. Evidently, $BR_{\bar{a},p}(x)$ is converging if and only if $\beta(1-p) < 1$ (and the limit is $\frac{\beta p \bar{a}}{1 - \beta(1-p)}$). In this case, $S^{\infty}_{\bar{a},p} = \left\{\frac{\beta p \bar{a}}{1-\beta(1-p)}\right\}$ and $\overline{a} \notin S^{\infty}_{\bar{a},p}$: $\overline{a}$ is not a $p-$consensus outcome for a value $p > 1 - \frac{1}{\beta}$. When $p \leq 1 - \frac{1}{\beta}$ (*i.e.*, $1 \leq \beta(1-p)$), then there are three cases to consider:

(a) If $\bar{a} > \frac{\beta(1-p)-1}{\beta p}$ $(BR_{\bar{a},p}(-1) > -1)$, then 1 is the unique fixed point, $BR_{\bar{a},p}(a) \geq a$ for every a. Any sequence $BR^n_{\bar{a},p}(a)$ converges to 1. Hence, $S^{\infty}_{\bar{a},p} = \{1\}$.

(b) If $\bar{a} < \frac{1-\beta(1-p)}{\beta p}$ $(BR_{\bar{a},p}(1) < 1)$, then -1 is the unique fixed point, $BR_{\bar{a},p}(a) \leq a$ for every a. Any sequence $BR^n_{\bar{a},p}(a)$ converges to -1. Hence, $S^{\infty}_{\bar{a},p} = \{-1\}$.

(c) If $\frac{\beta(1-p)-1}{\beta p} \geq \bar{a} \geq \frac{1-\beta(1-p)}{\beta p}$ $(BR_{\bar{a},p}(-1) = -1$ and $BR_{\bar{a},p}(1) = 1)$, then $BR_{\bar{a},p}$ admits 3 fixed points (equal to -1, $\frac{\beta p\bar{a}}{1-\beta(1-p)}$ and 1). For $a \leq \frac{\beta p\bar{a}}{1-\beta(1-p)}$, any sequence $BR_{\bar{a},p}(a)$ converges to -1. For $a \geq \frac{\beta p\bar{a}}{1-\beta(1-p)}$, any sequence $BR_{\bar{a},p}(a)$ converges to 1. Hence, $S^{\infty}_{\bar{a},p} = [-1,1]$. Therefore, $\bar{a} \in S^{\infty}_{\bar{a},p}$ iff $\frac{\beta(1-p)-1}{\beta p} \geq \bar{a} \geq \frac{1-\beta(1-p)}{\beta p}$ or equivalently $p \leq \frac{1}{1+\bar{a}}\left(1-\frac{1}{\beta}\right)$ and $p \leq \frac{1}{1-\bar{a}}\left(1-\frac{1}{\beta}\right)$.■

This following diagram illustrates the preceding corollary:



$p-$consensus in the case $\beta > 1$ (example with $\beta = 3$)

3 The concepts and key properties

We begin by describing the underlying strategic framework. We define the two main concepts we develop here, $p-$consensus and $p-$stability and provide a characterization of their basic properties.

3.1 The Model

The underlying model is due to MasColell (1984). Let A be a non-empty compact metric space of actions and $\Delta(A)$ be the compact and metrizable set of Borel probability measures on A endowed with the weak convergence topology which is metrizable using the Prohorov metric. For later reference, we note that the Prohorov metric is defined by:

$$d_{\Delta(A)}(m, \tau_a^*) = \inf\{\varepsilon > 0 : m(M) \leq \tau_a^*(M^\varepsilon) + \varepsilon \text{ for all Borel subsets } M \text{ of } A\},$$

where $M^\varepsilon = \{y \in A / d_A(x, y) \leq \varepsilon \text{ for some } x \in M\}$.[2] Let U_A be the set of continuous utility functions $u : A \times \Delta(A) \rightarrow R$ endowed with the supremum norm, the metric, separable and complete space of player characteristics. A game with a continuum of players is a Borel measure μ on U_A. For any probability measure τ on $U_A \times A$, let τ_u and τ_a denote the respective marginal distribution on U_A and A respectively. Let T denote the set of probability measures on $U_A \times A$ such that $\tau_u = \mu$, $\tau \in T$: T denotes the set of "strategy profiles", *i.e.*, a distribution of actions for each u. For $\tau \in T$, let $B_\tau = \{(u, a) : u(a, \tau_a) \geq u(A, \tau_a)\}$. The best-response correspondence is a map $\phi : T \rightarrow T$ such that $\phi(\tau) = \{\tau' \in T : \tau'(B_\tau) = 1\}$, *i.e.*, ϕ is the set of "strategy profiles" putting probability one on the fact that each player plays a best-response to τ. A Nash equilibrium is a measure $\tau^* \in T$ such that $\tau^* \in \phi(\tau^*)$.

Existence result (Theorem 1 in MasColell (1984)). For a given μ, there exists a Nash equilibrium distribution τ^*.

Remark: The above framework requires all agents who have the same utility function must also choose the same actions and/or have the same the beliefs over distributions. On the face of it, in an example where all agents have the same utility functions, this would require agents to have "homogeneous" beliefs. But, by re-interpreting the framework so that utility functions are differ up to an additive constant, we can use the above framework in setting where agents with the same utility functions have heterogeneous beliefs.

3.2 p-consensus and p-stability: definition

For a fixed τ and $p \in [0, 1]$, a $p-$belief is a probability distribution $\tau_p = p\tau + (1-p)\tau'$, for some $\tau' \in T$ *i.e.*, a belief that assigns a probability p to the distribution τ. Let $T_{\tau,p} \subseteq T$ denote the corresponding set.

We define a $p-$consensus distribution iteratively as follows. Let $S^0_{\tau,p} = T_{\tau,p}$ and consider the sequence of sets $S^n_{\tau,p}(\tau) = \left[\phi(S^{n-1}_{\tau,p})\right] \cap S^{n-1}_{\tau,p}$ for $n \geq 1$. This sequence is decreasing and therefore, it converges to a set $S^\infty_{\tau,p}$. Then, τ is a $p-$consensus distribution if $\tau \in S^\infty_{\tau,p}$.

For any $p' < p$, $p, p' \in [0, 1]$, we have that $S^n_{\tau,p} \subseteq S^n_{\tau,p'}$ as $T_{\tau,p} \subseteq T_{\tau,p'}$. Therefore, $S^\infty_{\tau,p} \subseteq S^\infty_{\tau,p'}$ and if $\tau \in S^\infty_{\tau,p}$ then $\tau \in S^\infty_{\tau,p'}$. It follows only a rationalizable distribution can be a $p-$consensus distribution and for a rationalizable

[2] See Dudley (1989) for this definition and other properties of the Prohorov metric not explicitly mentioned in this paper.

distribution, the set $I_\tau = \{p \in [0,1] : \tau \in S^\infty_{\tau,p}\}$ is either empty or $[0,p]$ for some $p \in [0,1]$.

Definition 1. When I_τ is non-empty, we say τ is a $p-$consensus distribution for $p = \sup I_\tau$.

The standard definition of rationalizability in this set-up would be to require $\tau \in S^\infty_{\tau,0}$ (note that $S^\infty_{\tau,0}$ doesn't depend on the choice of τ). It follows that I_τ is non-empty iff τ is rationalizable. In this sense, $p-$consensus is a refinement of rationalizability and the interesting question is whether τ is a $p-$consensus distribution for $p \neq 0$.

When $p = 1$, the set of $1-$consensus distributions and Nash equilibrium distributions must coincide.

We define a Nash equilibrium τ^* to be $p-$stable if the equilibrium distribution is the only element surviving the iterated elimination of non best-responses to a $p'-$belief for all $p' > p$. This definition relies on a "standard" definition of rationalizable outcomes in a game where the strategy set is restricted to $T_{\tau^*,p}$: a $p-$stable equilibrium is an equilibrium that is the only rationalizable outcome in a game with the restricted strategy set $T_{\tau^*,p}$.

For any $p < p'$, $p, p' \in [0,1]$, note that $S^\infty_{\tau^*,p'} \subseteq S^\infty_{\tau^*,p}$ and if $S^\infty_{\tau^*,p} = \{\tau^*\}$ then $S^\infty_{\tau^*,p'} = \{\tau^*\}$. In particular, the set $J_{\tau^*} = \{p \in [0,1] : S^\infty_{\tau^*,p} = \{\tau^*\}\}$ is an interval of the form $[p,1]$ for $p \in [0,1]$.

Definition 2. A Nash equilibrium τ^* is $p-$stable if $p = \inf J_{\tau^*} = \{p' \in [0,1] : S^\infty_{\tau^*,p'} = \{\tau^*\}\}$.

The interesting question is whether $p < 1$.

Remark:

1. We do not require that $S^\infty_{\tau^*,p} = \{\tau^*\}$ for a $p-$stable equilibrium. In some classes of games (for example, in the smooth one-dimensional case in section 3 below), $S^\infty_{\tau^*,p} \neq \{\tau^*\}$ at a $p-$stable equilibrium.

2. If τ^* is $0-$stable then τ^* is the unique rationalizable outcome.

3. If τ^* is $p'-$dominant then it follows that $S^1_{\tau^*,p'} = \{\tau^*\}$. Hence, $S^\infty_{\tau^*,p'} = \{\tau^*\}$ and therefore, τ^* is p-stable where $p \leq p'$.

3.3 Characterization of $p-$consensus and $p-$stability

The two definitions above bring added value in the analysis of strategic outcomes only when there are multiple rationalizable distributions. In what follows, this point is explored in greater detail.

We begin by providing an existence result for $p-$consensus distributions with $p \neq 0$.

Proposition 2. Suppose the set S^∞_0 of rationalizable distributions has a non-empty interior in T. Any distribution in the interior of S^∞_0 is $p-$ consensus distribution for some $p \neq 0$.

Proof. Consider $\widehat{\tau} \in Int.S^\infty_0$ and a small neighborhood $N \subset Int.S^\infty_0$ of $\widehat{\tau}$. For $\tau \in N$, by upper hemi-continuity of ϕ, there exists p close enough to zero and $\tau' \in S^\infty_0$ in a neighborhood N' of $\widehat{\tau}$ such that $\phi(p\widehat{\tau} + (1-p)\tau') = \tau$. Denote the corresponding set $B_{\widehat{\tau},\tau,p}$ and let $B_{\widehat{\tau},p} = \cup_{\tau \in S^\infty_0} B_{\widehat{\tau},\tau,p}$. Note that $B_{\widehat{\tau},p} \subseteq N'$. Let $T_{\widehat{\tau},p}(B) = \{p\widehat{\tau} + (1-p)\tau, \tau \in B\}$. Note that $T_{\widehat{\tau},p}(B_{\widehat{\tau},p}) \subseteq Int.T_{\widehat{\tau},p}$. Hence, for any τ in N, for each N small enough, $T_{\widehat{\tau},p}(B_{\widehat{\tau},p}) \subseteq Int.T_{\tau,p}$, so that

$T_{\widehat{\tau},p}(B_{\widehat{\tau},p}) = T_{\tau,p}(B_{\tau,p})$; moreover, by upper hemi-continuity of ϕ, $B_{\tau,p} \subseteq S_0^\infty$ and hence, $N \subseteq S_0^\infty$, as required.■

Next, we examine the conditions under which an equilibrium is $p-$stable. To this end, define a best-response correspondence $B(u,m) = \{a \in A : u(a,m) \geq u(A,m)\}$: an action in $B(u,m)$ is a best-response for $u \in U_A$ to some $m \in \Delta(A)$.

For each $m \in \Delta(A)$, consider the set

$$\tilde{U}_A(m) = \left\{ \begin{array}{c} u \in U_A : B(u,m) \text{ is not single-valued or} \\ \limsup_{m' \to m} \frac{d_A(B(u,m'),B(u,m))}{d_{\Delta(A)}(m',m)} < \infty \end{array} \right\},$$

where d_A denotes a distance on A and $d_{\Delta(A)}(.,.)$ denotes the Prohorov metric on $\Delta(A)$. Consider two Dirac measures δ_x and δ_y. Then, $d_\Delta(A)(\delta_x,\delta_y) = d_A(x,y)$. Note that for each $u \in \tilde{U}_A(m)$, a small change in m induces an infinitely large change in best-responses.

Consider a given τ^*. For every u, let

$$k_{u,\tau} = \lim \sup_{m \to \tau_a} \frac{d_A(B(u,m),B(u,\tau_a))}{d_{\Delta(A)}(m,\tau_a)},$$

and

$$K_\tau = \sup_{u \in U_A} \mathtt{ess} k_{u,\tau}.$$

K_τ is the essential upper bound of $k_{u,\tau}$ w.r.t. measure μ (that is: the set of u such that $k_{u,\tau} > K_\tau$ has $\mu-$measure 0).

Definition 3. The equilibrium τ^* of a game μ is admissible if $\mu\left(\tilde{U}_A(\tau_a^*)\right) = 0$ and

$$K_\tau < +\infty.$$

As a preliminary step, the following lemma[3] summarizes three key properties of the Prohorov metric that will be useful in proving the result below.

Lemma 1. (i) Consider $\tau = p\widehat{\tau} + (1-p)\tau'$. Then,

$$d_{\Delta(A)}(\tau_a,\widehat{\tau}_a) \leq (1-p)\, d_{\Delta(A)}(\tau'_a,\widehat{\tau}_a).$$

(ii) Consider a Dirac measure δ_x and a distribution $\tau_a \in \Delta(A)$. Consider S the support of τ_a (the smallest closed set s.t. $\tau_a(S) = 1$) and $d = \sup_{y \in S} d_A(x,y)$ (d is the radius of the smallest ball centered on x that contains S)[4]. Then,

$$d_{\Delta(A)}(\delta_x,\tau_a) \leq d.$$

(iii) Consider $\tau_a \in \Delta(A)$ defined by $\tau_a = \int \tau_\lambda f(d\lambda)$ where f is a probability distribution on a set of parameters λ. Consider another distribution $\nu \in \Delta(A)$. We have

$$d_{\Delta(A)}(\tau_a,\nu) \leq \sup_\lambda \mathtt{ess} d_{\Delta(A)}(\tau_\lambda,\nu).$$

[3] We state and prove this lemma for completeness as we are not aware of an explicit proof of the three properties of the Prohorov metric contained in the lemma and these required for the proof of Proposition 1 below.

[4] Notice that x may be in S or not.

Proof. See appendix.■

We are now in a position to state and prove the following result:

Proposition 3. For any admissible equilibrium, there is a $\hat{p} < 1$ such that the equilibrium is $\hat{p}-$stable.

Proof. See appendix.■

Heuristically, the idea underlying the proof is as follows. An equilibrium τ^* is $p-$stable if the best-response map, restricted to $p-$beliefs, generating the sequence of sets $S^n_{\tau^*,p}$ is a contraction. For p close to one, when the equilibrium τ^* is admissible, we show that the best-response map, restricted to $p-$beliefs, cannot vary much (i.e. in the smooth case, the derivative of the best-response map, restricted to $p-$beliefs, is small). If, on the contrary, the equilibrium τ^* isn't admissible, even when restricted to $p-$beliefs, it can change dramatically around the equilibrium implying that the preceding step of the argument doesn't hold.

We conclude our characterization by proving a result linking $p-$consensus and $p-$stability.

Proposition 4. Consider a $p-$stable equilibrium τ^*.

(i) Suppose K_τ is continuous in τ. Then, for any $\hat{p} > p$, there is a neighborhood of τ^* such that no τ belonging to the neighborhood is a $p'-$consensus distribution for a value $p' \geq \hat{p}$.

(ii) Suppose $S^\infty_{\tau^*,p}$ has a non-empty interior in T and ϕ is a continuous correspondence. Then, for any $\hat{p} < p$, there exists a $\tau \in S^\infty_{\tau^*,p}$ arbitrarily close to τ^* such that τ is a $p'-$consensus distribution for some $p' \geq \hat{p}$.

Proof. (i) The proof relies on the proof of Proposition 3 where it is shown that a necessary condition for $p-$stability is $K_{\tau^*}(1-p) \leq 1$; hence, $K_{\tau^*}(1-\hat{p}) < 1$. By the smoothness assumption, it follows that for τ arbitrarily close to τ^*, $K_\tau(1-\hat{p}) < 1$. So $S^\infty_{\tau,\hat{p}}$ is either empty of has a radius 0; and $\tau \notin S^\infty_{\tau,\hat{p}}$ given that τ is not a Nash distribution. Since $p' \geq \hat{p}$ implies $S^\infty_{\tau,p'} \subset S^\infty_{\tau,\hat{p}}$, the result follows.

(ii) Consider an open neighborhood N of τ^* in $S^\infty_{\tau^*,p}$. By continuity of ϕ and definition of $S^\infty_{\tau^*,p}$, we have that $S^\infty_{\tau^*,p} = \phi\left(S^\infty_{\tau^*,p}\right) \cap T_{\tau^*,p}$. This means that for every $\tau \in N$, there exists a non empty set $B_{\tau^*,p}(\tau)$ of distributions $\tau' \in S^\infty_{\tau^*,p}$ such that $\phi(p\tau^* + (1-p')\tau') = \tau$. Let $B_{\tau^*,p} = \cup_{\tau \in N} B_{\tau^*,p}(\tau)$. Note that $B_{\tau^*,p} \subset S^\infty_{\tau^*,p}$ and $\tau^* \in B_{\tau^*,p}$ (since $\tau^* \in B_{\tau^*,p}(\tau^*)$). Let $T_{\tau^*,p}(N) = \{p\tau^* + (1-p)\tau', \tau' \in N\}$. Note that $T_{\tau^*,p}(B_{\tau^*,p}) \subseteq Int.T_{\tau^*,p}$; hence, for any τ in N (provided N is chosen small enough), $T_{\tau^*,p}(B_{\tau^*,p}) \subseteq Int.T_{\tau,p}$. Therefore, $T_{\tau^*,p}(B_{\tau^*,p}) = T_{\tau,p}(B_{\tau,p})$ so that, by continuity of ϕ, $B_{\tau,p} \subseteq S^\infty_{\tau^*,p}$ so that $N \subseteq S^\infty_{\tau,p}$, as required.■

An informal interpretation of Point (i) is that under a mild continuity restriction (which would typically be satisfied, for example, in smooth settings (see Section 3.2 below)), a $p-$stable equilibrium is a locally isolated $p-$consensus distribution, *i.e.* there is no $p'-$consensus distribution in its vicinity with a degree of consensus p' larger than p. Informally again, point (ii) says the converse statement also holds provided the best-response map ϕ is continuous: in the vicinity of a $p-$stable equilibrium, there is always at least one $p'-$consensus distribution with a degree of consensus p' smaller than p and arbitrarily close to p (provided that the interior of $S^{\infty}_{\tau^*,p}$ is not empty).

However, note that these informal interpretations are offered to aid understanding of Proposition 4; the analogous formal statements cannot, in general, be proved. Indeed, the (very simple) game developed in Section 3.1 below is a counterexample: there is a $p-$stable equilibrium such that every distribution τ in its neighborhood is a $p_\tau-$consensus distribution for some value $p_\tau > p$ (p_τ depends on τ); there is another $p-$stable equilibrium with no $p_\tau-$consensus distribution in its neighborhood satisfying $p_\tau \geq p$. The point is that p_τ tends to p when τ tends to the equilibrium.

Lastly, the requirement that $S^{\infty}_{\tau^*,p}$ has a non-empty interior in T is a restriction that is key to Proposition 4 and may not always be satisfied when the action set has at least two dimensions. To see this point, consider the simple case of a game where an action is a vector in $\Re^K$ for some $K \geq 2$ and the utility depends only on the individual action and the average action (*i.e.* only the first moment of the distribution of actions matter for the strategic interaction). The notation $S^{\infty}_{\tau,p}$ and τ are easily redefined as sets and elements in $\Re^K$. In such a game, it may be the case that $S^{\infty}_{\tau^*,p}$ has a dimension strictly less than K (hence, an empty interior). For τ close to τ^*, the set $S^{\infty}_{\tau,p}$ may have a dimension strictly less than K as well (it may be close to $S^{\infty}_{\tau^*,p}$ by a continuity argument). Because of the low dimension of $S^{\infty}_{\tau,p}$, it is fully possible that τ is close to τ^*, $S^{\infty}_{\tau,p}$ is close to $S^{\infty}_{\tau^*,p}$ and yet $\tau \notin S^{\infty}_{\tau,p}$. Consequently, there may be no τ in the vicinity of τ^* that are $p-$consensus.

3.4 p-stability in the smooth one-dimensional case

To obtain an intuitive feel for the notion of stability being studied in this paper, we extend the piecewise linear setting of the previous subsection to a non linear setting.

Consider a simple, smooth model of strategic interaction where there is a continuum of agents each whom chooses an action $a \in A$ (a compact set in R) to maximize $u(a, \bar{a})$ (C^2, with $u''_{aa} < 0$) where $\bar{a}$ is the average action. Without loss of generality, $A = [-1, 1]$. Suppose, there is a (not necessarily) unique Nash that is interior and is normalized to 0 so that $u'_a(0,0) = 0$. Denote $BR(\bar{a})$ the (unique) best response to $\bar{a}$ (characterized by $u'_a(BR(\bar{a}), \bar{a}) = 0$). We assume that the BR map is not vertical at equilibrium ($BR'(0) < +\infty$).

We are now in a position to state the following result:

Proposition 5. There is $\hat{p} < 1$ such that the equilibrium is $\hat{p}$-stable. If $\sup_{\bar{a}\in[-1,1]} |BR'(\bar{a})| < 1$, then $\hat{p} = 0$. Otherwise, we have:

$$1 - \frac{1}{|BR'(0)|} \leq \hat{p} \leq 1 - \frac{1}{1 + (M-1)\,m}, \tag{1}$$

where

$$m = \sup_{a,\bar{a}\in[-1,1]} \frac{u''_{a\bar{a}}(a,\bar{a})}{u''_{a\bar{a}}(a,0)} \geq 1,$$

$$M = \sup_{a,\bar{a}\in[-1,1]} \left| \frac{u''_{a\bar{a}}(a,\bar{a})}{u''_{aa}(a,\bar{a})} \right| \geq 1.$$

For any $p < \hat{p}$, there exists a neighborhood of 0 such that every action in this neighborhood is the average action of a $p-$consensus distribution. For any $p > \hat{p}$, there exists a neighborhood of 0 such that no action in this neighborhood is the average action of a $p-$consensus distribution.

Proof. See appendix.■

Notice that by implicit functions theorem, $|BR'(\bar{a})| \leq M$ for any $\bar{a}$. To relate $\hat{p}$ with exogenous variables, rewrite the left inequality (1) using $BR'(0) = -u''_{a\bar{a}}(0,0)/u''_{aa}(0,0)$ (by implicit functions theorem, again). The linear case developed in the previous subsection corresponds to the case with a quadratic utility: $m = 1$, BR' is constant (equal to M) and $\hat{p} = 1 - 1/M$. As in the linear case, our stability concept gives a motivation for looking at the slope of the best response map as a stability index.

A special case of the model studied so far is the Muth model with a large number of farmers who have to commit to an output level before selling their products in a competitive market in Guesnerie (1992). Farmer i maximizes $\pi q - \frac{q^2}{2C^i}$ (π is the output price). Aggregate supply in this market is given by $S(\pi) = C\pi$ where $C = \int C^i di$. Aggregate demand in this market is:

$$D(\pi) = \begin{cases} A - B\pi \ if \ \pi \leq \frac{A}{B} \\ 0, \ otherwise \end{cases}$$

Let π^* be the competitive equilibrium price. Guesnerie (1992) shows that when the slope of the best response map $B/C < 1$, π^* is the unique rationalizable outcome.

Applying Proposition 5 immediately yields that the equilibrium in Guesnerie's model is $\hat{p}-$stable for $\hat{p} = \max\{1 - C/B, 0\}$. Thus, $p-$stability describes more precisely the degree of stability of the equilibrium when it is not the unique rationalizable outcome.

The remainder of the section is devoted to the proof of Proposition 5. The proof shows that $p-$stability relies on the best response map $BR_{0,p}(\bar{a})$ (best response to beliefs "probability p on 0, probability $(1-p)$ on $\bar{a}$"). When p is close to one, the slope $BR'_{0,p}(\bar{a})$ is small enough (whatever $\bar{a}$ is). The map $BR_{0,p}$ is then globally contracting and $p-$stability obtains. Intuitively, when p is close to one, the best response is not very sensible to the value $\bar{a}$ and the best response cannot deviate very much from the equilibrium value 0. This is the condition needed to get $p-$stability.

3.5 Basin of attraction and p-stability

In this subsection, in an example with three equilibria, we show that the $p-$stability of a corner equilibrium is linked to the slope of the aggregate best response at that equilibrium but not to the size of the basin of attraction.

We consider a game with a continuum of individuals of mass one who must choose an action $a \in [-1,1]$ to maximize the payoffs

$$-\frac{a^2}{2} + aP(\overline{a})$$

where $\overline{a}$ denotes the average action and $P(.)$ is a third degree polynomial specified below. At an interior best response, the first order condition is $P(\overline{a})-a=0$.

Assume that $P(\overline{a}) = -A(\overline{a}-1)(\overline{a}+1)(\overline{a}-\alpha)$, where $A>0$ and $\alpha \in (-1,1)$. Then, there are three equilibria (fix points): $\overline{a}=1$, $\overline{a}=-1$, $\overline{a}=\alpha$. By computation, we can check that

$$\left.\frac{\partial P(\overline{a})}{\partial \overline{a}}\right|_{\overline{a}=1} = 1-2A(1-\alpha), \left.\frac{\partial P(\overline{a})}{\partial \overline{a}}\right|_{\overline{a}=-1} = 1-2A(1+\alpha), \left.\frac{\partial P(\overline{a})}{\partial \overline{a}}\right|_{\overline{a}=\alpha} = 1+A(1-\alpha^2).$$

Note that $\left.\frac{\partial P(\overline{a})}{\partial \overline{a}}\right|_{\overline{a}=\alpha} > 1$ while $\left.\frac{\partial P(\overline{a})}{\partial \overline{a}}\right|_{\overline{a}=1}$ and $\left.\frac{\partial P(\overline{a})}{\partial \overline{a}}\right|_{\overline{a}=-1}$ are both less than one. To ensure that we are in the case of strategic complements we restrict the parameters so that both $\left.\frac{\partial P(\overline{a})}{\partial \overline{a}}\right|_{\overline{a}=1} > 0$ and $\left.\frac{\partial P(\overline{a})}{\partial \overline{a}}\right|_{\overline{a}=-1} > 0$. Therefore, $\overline{a}=1$ (respectively, $\overline{a}=-1$) is stable in the best-response dynamics on its basin of attraction $(\alpha,1]$ (respectively, $[-1,\alpha)$). Furthermore, it follows that the whole action set is rationalizable so, in particular, none of the three equilibria is 0-stable.

Next, we show that by choosing different value of A we can choose different values of α consistent with the $p-$stability of $\overline{a}=1$. By computation, the $p-$best response map is

$$pP(\overline{a})+(1-p)P(\overline{a}') = pP(\overline{a}) - A\left[+(1-p)(\overline{a}'-1)(\overline{a}'+1)(\overline{a}'-\alpha)\right] + (1-p)\overline{a}'$$

Observe that $p-$stability requires the above best-response map to be convergent and we need a unique fix point of the preceding map. So we look for value of p for which the preceding $p-$best response has exactly one root. So the requiring $p-$stability of either one of the two equilibria for all $p>\overline{p}$ requires us to calculate the value of p for which the preceding $p-$best response has exactly two roots one of which is the equilibrium and the other one is the double root. So this implies

$$pP(\overline{a})+(1-p)P(\overline{a}') = -A(1-p)(\overline{a}'-\overline{a})(\overline{a}'-\delta)^2$$

where δ is the unknown double root.

Consider the $p-$stability of $\overline{x}=1$. Identifying the coefficients (the coefficients of degree 3 are identical)

$$\begin{aligned}(1-p)\,\alpha A &= (1-p)\,A\,(1+2\delta)\\ (1-p)\,(1+A) &= 1-(1-p)\,A\,(2+\delta)\,\delta\\ p-(1-p)\,\alpha A &= (1-p)\,A\delta^2\end{aligned}$$

The first one requires $\delta=\frac{\alpha-1}{2}$ while the second one requires

$$(1-p)\,A\,(1+\delta)^2=p$$

which is $\delta=\sqrt{\frac{p}{(1-p)A}}-1$ (the second one follows from the others). Hence, p is such that

$$\begin{aligned}\frac{\alpha-1}{2} &= \sqrt{\frac{p}{(1-p)\,A}}-1\\ \frac{A\,(\alpha+1)^2}{4} &= \frac{p}{(1-p)}\end{aligned}$$

Observe that we can rewrite the preceding expression as

$$\frac{\left(1-\left.\frac{\partial P(\overline{a})}{\partial\overline{a}}\right|_{\overline{a}=1}\right)(\alpha+1)^2}{8(1-\alpha)}=\frac{p}{(1-p)}$$

It follows that by choosing different value of A we can choose different values of α consistent with the $p-$stability of $\overline{a}=1$.

Therefore, there is no link between $p-$stability of $\overline{a}=1$ and the size of its basin of attraction.

4 Intertemporal trade, expectations coordination and bubbles

In this section, we study the partial consensus outcomes in a two period economy with a single asset linking the two periods. The aim is to examine the foundations, via belief coordination, of perfect foresight equilibria. We are in a setting where all agents are price-takers and payoffs depend on their own actions and market prices; hence, we adopt a slightly different formalization of a large economy from MasColell (1984). The analysis generalizes and extends Ghosal (2006)'s local stability analysis of a perfect foresight equilibrium. In addition, to a new solution concept for intertemporal economies being proposed and its links with perfect foresight equilibria being analyzed, we allow preferences to be non-separable over time.

4.1 The Economy

The economy consists of a mass of individuals of finite measure, formally, an atomless measure space of individuals, $\{I, \iota, \mu\}$, with I the set of agents, ι the σ-algebra on I and μ an atomless measure defined on I. Null sets of individuals are systematically ignored throughout the paper. For some arbitrary finite $K-$dimensional Euclidian space, an assignment is any function $\mathbf{g} : I \to \Re^K$ each coordinate of which is integrable[5].Trade in this economy is sequential and takes place at two time periods, $t = 1, 2$, with L_t, $t = 1, 2$, commodities traded in the spot commodity markets in period t[6] and an asset market that opens in the first period. In time period $t = 1$, each individual submits commodity demands in the spot commodity markets and asset demands in the asset market. Prices in these markets then adjust to ensure market clearing. In time period $t = 2$, each individual submits commodity demands in the spot commodity markets. Prices in these markets then adjust to ensure market clearing.

A commodity bundle is $x \in \Re_+^{L_1} \times \Re_+^{L_2}$ with x_{tl} denoting quantities of consumption of commodity l in period t. Endowments are $\mathbf{w} : I \to \Re_+^{L_1} \times \Re_+^{L_2}$ with $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$, and $\bar{\mathbf{w}}_t = \int \mathbf{w}_t^i di \gg 0$ for $t = 1, 2$. The asset traded in the first period pays off in units of the first commodity traded in the second period and further, it is in zero net supply. Preferences of trader i are described by a utility function $u^i : \Re_+^{L_1} \times \Re_+^{L_2} \to \Re$ such that two assumptions are satisfied: $(A1)$ For each traders $i \in I$, u^i satisfies strict monotonicity, strict concavity, is twice continuously differentiable on $\Re_+^{L_1} \times \Re_+^{L_2}$; $(A2)$ $u : I \times \Re_+^{L_1} \times \Re_+^{L_2} \to \Re$ is measurable and uniformly smooth. The requirement of uniform smoothness in $(A2)$ follows Aumann (1975, sections 4 and 10) except that we do not require the utility functions of any trader to be bounded. A consequence of $(A2)$ is that $u : I \times \Re_+^{L_1} \times \Re_+^{L_2} \to \Re$ viewed as a map from (i, x_1, x_2) to real numbers is measurable as a function of (i, x_1, x_2). The asset traded in the first period pays off in units of the first commodity traded in the second period and further, it is in zero net supply. An allocation is a triple $(\mathbf{x}_1, \mathbf{y}, \mathbf{x}_2)$ such that $\mathbf{x}_t^i \in \Re_+^{L_t}$, $t = 1, 2$, for all $i \in I$ and $\mathbf{y}^i \in \Re$. An allocation is feasible if, in addition, $\int \mathbf{x}_t^i di = \bar{\mathbf{x}}_t = \bar{\mathbf{w}}_t = \int \mathbf{w}_t^i di$, $t = 1, 2$ and $\bar{\mathbf{y}} = \int \mathbf{y}^i di = 0$. An economy is $\mathbf{E} = \{I, \iota, \mu, (u^i, w^i) : i \in I\}$.

Prices are (π_1, q, π_2) where π_{tl} is the spot commodity market price of commodity l in period t and q is the price of the asset. Normalize prices so that $\pi_{11} = 1$ and $\pi_{21} = 1$. As the utility function of each individual is strongly monotone, without loss of generality it is possible to restrict attention to prices where $\pi_t \in \Re_{++}^{L_t - 1}$, $t = 1, 2$ and $q \in \Re_{++}$. Asset payoffs are therefore denoted in the second period numeraire.

At prices (π_1, q, π_2) the maximization problem that each individual solves has two stages. In the second stage, at $t = 2$, given $(\pi_1, q, \pi_2, x_1, y)$ each individual

[5] Throughout this subsection, the bold face type will be used to denote an assignment, with the i^{th} component of the assignment $\mathbf{g}$ denoted by $\mathbf{g}^i$ and the k^{th} coordinate of the assignment $\mathbf{g}$ denoted by $\mathbf{g}_k$.

[6] Note that when $L_2 = 1$, with only one commodity at $t = 2$, the coordination problem in second period spot markets studied here disappears. Hence, we assume that $L_2 \geq 2$.

solves:

$$Max_{\{x_2\}}\ u^i(x_1, x_2)\ s.t.\ \pi_2 x_2 \leq \pi_2 \mathbf{w}_2^i + y,\ x_2 \in \Re_+^{L_2}$$

For a solution $\hat{x}_2^i$ to this maximization problem, let $v^i(x_1, \pi_2, \pi_2\mathbf{w}_2^i + y) = u^i(x_1, \hat{x}_2^i\left(x_1, \pi_2, \pi_2\mathbf{w}_2^i + y\right))$. In the first stage, at $t = 1$, given (π_1, q, π_2) each individual solves the following maximization problem:

$$Max_{\{x_1, y\}}\ v^i(x_1, \pi_2, \pi_2\mathbf{w}_2^i + y)\ s.t.\ \pi_1 x_1 + qy \leq \pi_1 \mathbf{w}_1^i,\ x_1 \in \Re_+^{L_1}$$

Let $(\hat{x}_1^i, \hat{y}^i)$ denote a solution to this sequential, two-stage maximization problem. Let $\hat{S}^i(\pi_1, q, \pi_2)$ denote the set of all possible solutions $(\hat{x}_1^i, \hat{y}^i, \hat{x}_2^i)$ at prices (π_1, q, π_2).

Definition 1 *A Perfect Foresight Equilibrium (PFE) is a vector of prices $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$ and allocations $(\hat{\mathbf{x}}_1, \hat{\mathbf{y}}, \hat{\mathbf{x}}_2)$ such that (a) at prices $(\hat{\pi}_1, q, \hat{\pi}_2)$, $(\hat{\mathbf{x}}_1^i, \hat{\mathbf{y}}^i, \hat{\mathbf{x}}_2^i) \in \hat{S}^i(\hat{\pi}_1, q, \hat{\pi}_2)$, for all $i \in I$, (b) $\hat{\mathbf{x}}_t \in \Re_{++}^{L_t}$, $t = 1, 2$, and (c) $\int \hat{\mathbf{x}}_t^i di = \bar{\mathbf{w}}_t$, $t = 1, 2$ and $\int \hat{\mathbf{y}}^i di = 0$.*

4.2 A local analysis of bubbles and preference heterogeneity

In this part of paper, we work with prices and allocations in the vicinity of a Perfect foresight equilibrium. This allows us to provide a reasonably complete characterization of the $p-$stability of a PFE and the condition under which, locally, bubbles exist.

The market clearing equations corresponding to a PFE are:

$$\int \hat{\mathbf{x}}_1^i(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)di = \bar{\mathbf{w}}_1, \int \hat{\mathbf{y}}^i(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)di = 0, \int \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1^i(\hat{\pi}_1, \hat{q}, \hat{\pi}_2), \hat{\pi}_2, \hat{\mathbf{y}}^i(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)) = \bar{\mathbf{w}}_2$$

Let $\bar{\mathbf{x}}_t = \int \hat{\mathbf{x}}_t^i di$, $t = 1, 2$ denote mean commodity demand in period t and $\bar{\mathbf{y}} = \int \hat{\mathbf{y}}^i di$ denote mean asset demand. Let $N(\hat{\pi}_1, \hat{q}, \hat{\pi}_2) \subset \Re_{++}^{L_1} \times \Re_{++} \times \Re_{++}^{L_2}$ be a neighborhood of an interior PFE price vector. Then, for all $(\pi_1, q, \pi_2) \in N(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$, the derivatives $\partial_{\pi_{t'l'}}\bar{\mathbf{x}}_{tl}$, $\partial_{\pi_{t'l'}}\bar{\mathbf{y}}$, $\partial_q \bar{\mathbf{x}}_{tl}$, $\partial_q \bar{\mathbf{y}}$ exist for all $t, t' = 1, 2$ and all $l, l' = 1, ..., L$ and are equal to $\partial_{\pi_{t'l'}}\bar{\mathbf{x}}_{tl} = \int \partial_{\pi_{t'l'}}\hat{\mathbf{x}}_{tl}^i di$, $\partial_{\pi_{t'l'}}\bar{\mathbf{y}} = \int \partial_{p_{t'l'}}\hat{\mathbf{y}}^i di$, $\partial_q \bar{\mathbf{x}}_{tl} = \int \partial_q \hat{\mathbf{x}}_{tl}^i di$, $\partial_q \bar{\mathbf{y}} = \int \partial_q \hat{\mathbf{y}}^i di$. This follows from the fact that $\partial_{\pi_{t'l'}}\hat{\mathbf{x}}_{tl}^i$, $\partial_{\pi_{t'l'}}\hat{\mathbf{y}}^i$, $\partial_q \hat{\mathbf{x}}_{tl}^i$, $\partial_q \hat{\mathbf{y}}^i$, for all $l, l' = 1, ..., L_t$, $t, t' = 1, 2,$, $i \in I$ are integrally bounded (see, for instance, page 154, Jones (1993)).

After deleting the numeraire commodity in each period, consider the Jacobian of the market clearing equations $J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$ where $J_{11} = \begin{pmatrix} \partial_{\pi_1}\bar{\mathbf{x}}_1 & \partial_q \bar{\mathbf{x}}_1 \\ \partial_{\pi_1}\bar{\mathbf{y}} & \partial_q \bar{\mathbf{y}} \end{pmatrix}$, $J_{12} = \begin{pmatrix} \partial_{\pi_2}\bar{\mathbf{x}}_1 \\ \partial_{\pi_2}\bar{\mathbf{y}} \end{pmatrix}$, $J_{21} = \begin{pmatrix} \partial_{\pi_1}\bar{\mathbf{x}}_2 & \partial_q \bar{\mathbf{x}}_2 \end{pmatrix}$, $J_{22} = (\partial_{\pi_2}\bar{\mathbf{x}}_2)$ evaluated at the market clearing prices $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2) \in \Re_{++}^{L_1 - 1} \times \Re_{++} \times \Re_{++}^{L_2 - 1}$, where the numeraire commodity in each period has been deleted as well. For any assignment $\mathbf{x}_1, \mathbf{y}$ such that $\int \mathbf{x}_1^i di = \bar{\mathbf{w}}_1$, $\int \mathbf{y}^i di = 0$, let $\partial_{\pi_2}\bar{\mathbf{x}}_2(\mathbf{x}_1, \mathbf{y}) = \int \partial_{\pi_2}\hat{\mathbf{x}}_2^i(\mathbf{x}_1^i, p_2, \mathbf{y}^i)$.

Definition 2 *(Regularity, Strong Regularity and Sequential Regularity)*[7] *An interior PFE is regular (respectively, strongly regular) if J is invertible (respectively, J_{11} and J_{22} are invertible). It is sequentially regular if, in addition to being regular and strongly regular, $\left(\partial_{\pi_2}\bar{\mathbf{x}}_2(\mathbf{x}_1,\mathbf{y})\right)^{-1}$ exists for all assignment of assets $\mathbf{y}$ such that $\mathbf{y}^i = \hat{\mathbf{y}}^i(\pi_1', q', \pi_2')$ for all $i \in I$ and some $(\pi_1', q', \pi_2') \in N(\hat{\pi}_1, \hat{q}, \hat{\pi}_2, \varepsilon)$. An economy is regular (respectively, strongly regular and sequentially regular) if all its interior PFE are regular (respectively, strongly regular and sequentially regular).*

In a regular economy, each PFE is locally isolated. In a strongly regular economy, in addition, it follows as a consequence of the implicit function theorem that for a given $(\bar{\pi}_1, \bar{q}) \in N(\hat{\pi}_1, \hat{q},, \varepsilon) \subset \Re^{L_2-1}_{++} \times \Re$, there is a locally unique second period price π_2 that solves $\int \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1^i(\bar{\pi}_1, \bar{q}, \pi_2), p_2, \hat{\mathbf{y}}^i(\bar{\pi}_1, \bar{q}, p_2)) = \bar{\mathbf{w}}_2$; further, it is a continuous function of $(\bar{\pi}_1, \bar{q},) \in N(\hat{\pi}_1, \hat{q}, \varepsilon)$. Moreover, in a sequentially regular economy, again as a consequence of the implicit function theorem, for all $(\pi_1', q', \pi_2') \in N(\hat{\pi}_1, \hat{q}, \hat{\pi}_2, \varepsilon)$ with $\int \hat{\mathbf{x}}_1^i(\pi_1', q', \pi_2')di = \bar{\mathbf{w}}_1$ and $\int \hat{\mathbf{y}}^i(\pi_1', q', \pi_2')di = 0$, there exists a unique second period price π_2 that solves $\int \hat{\mathbf{x}}_2^i(\pi_1', q', \pi_2'), \pi_2, \hat{\mathbf{y}}^i(\pi_1', q', \pi_2')) = \bar{\mathbf{w}}_2$; further, it is a continuous function of $(\pi_1', q', \pi_2') \in N(\hat{\pi}_1, \hat{q}, \hat{\pi}_2, \varepsilon)$.

We begin our analysis assuming that individuals have expectations over future prices whose support is some set[8] $\Pi_2^0 \subset\subseteq N(\hat{\pi}_2, \varepsilon)$. Consider $\widetilde{\pi}_2 \in \Pi_2^0$. A p-belief puts a weight p on $\widetilde{\pi}_2$ and a weight $1-p$ on $\pi_2^{e,i} \in \Pi_2^0$ (not assumed to be common knowledge). The interpretation is that there is partial consensus (where p measures the degree of consensus) on $\widetilde{\pi}_2$. Let $\mathbf{f} : I \to \Pi_2^0 \subset \Re^{L_2-1}_{++}$: an assignment of expectations where $\mathbf{f}^i = \pi_2^{e,i}$. Let $\hat{\mathbf{x}}_2^i\left(x_1, \widetilde{\pi}_2, \widetilde{\pi}_2\mathbf{w}_2^i + y\right)$ denote the (unique) solution to

$$Max_{\{x_2\}}\ u^i(x_1, x_2)\ s.t.\ \widetilde{\pi}_2 x_2 \leq \widetilde{\pi}_2\mathbf{w}_2^i + y,\ x_2 \in \Re^{L_2}_{+}$$

Let $\hat{\mathbf{x}}_2^i\left(x_1, \mathbf{f}^i, \mathbf{f}^i\mathbf{w}_2^i + y)\right)$ denote the (unique) solution to

$$Max_{\{x_2\}}\ u^i(x_1, x_2)\ s.t.\ \mathbf{f}^i x_2 \leq \mathbf{f}^i\mathbf{w}_2^i + y,\ x_2 \in \Re^{L_2}_{+}$$

For each $p-belief$, there is an associated lottery over period 2 consumption l_p^i, with probability p on $\hat{\mathbf{x}}_2^i\left(x_1, \widetilde{\pi}_2, \widetilde{\pi}_2\mathbf{w}_2^i + y\right)$ and with probability $1-p$ on $\hat{\mathbf{x}}_2^i\left(x_1, \mathbf{f}^i, \mathbf{f}^i\mathbf{w}_2^i + y\right)$. Let $\mathbf{l}_p$ denote an assignment of lotteries. At $t = 1$, given (π_1, q) and l_r^i an individual solves:

$$Max_{\{x_1,y\}}\ pv^i\left(x_1, \widetilde{\pi}_2, \widetilde{\pi}_2\mathbf{w}_2^i + y\right) + (1-p)v^i\left(x_1, \mathbf{f}^i, \mathbf{f}^i\mathbf{w}_2^i + y\right)]\ s.t.\ \pi_1 x_1 + qy \leq \pi_1\mathbf{w}_1^i,\ x_1 \in \Re^{L_1}_{+}.$$

Let $\hat{\mathbf{x}}_1^i(\pi_1, q, \mathbf{l}_p^i)$, $\hat{\mathbf{y}}^i(\pi_1', q', \mathbf{l}_p^i)$ denote a solution to the preceding sequential, two-stage maximization problem.

[7] Sequential regularity was introduced by Balasko (1994).

[8] Our analysis, in this subsection is in the vicinity of a PFE; hence, we ignore any issues that arise with bankruptcy in the initial set of prices Π_2^0. In a global analysis, as in the subsection below, bankruptcy constraints will be explicitly incorporated in the characterization of Π_2^0.

For a fixed $\mathbf{l}_p$, (π_1', q') is a *period 1 equilibrium* if and only if $\int \hat{\mathbf{x}}_1^i(\pi_1', q', \mathbf{l}_p^i)di = \int \mathbf{w}_1^i di$ and $\int \hat{\mathbf{y}}^i((\pi_1', q', \mathbf{l}_p^i)di = 0$. Let $\hat{E}_1(\mathbf{l}_p)$ denote the set of such equilibria. For a pair (π_1, q) first period prices, with a mild abuse of notation, the second period price π_2' is a *period 2 equilibrium* if and only if $\int \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1^i, \pi_2', \hat{\mathbf{y}}^i(\pi_1, q, \mathbf{l}_p^i)) = \int \mathbf{w}_2^i di$. Let $\hat{E}_2(\hat{\mathbf{x}}_1, \pi_1, q, \mathbf{l}_p)$ denote the set of such equilibria.

Fix $\Pi_2^0 \subset \Re_{++}^{L_2-1}$. For $n = 1, ...,$ define

$$\Pi_{2,p}^n = \left\{ \begin{array}{c} \pi_2 \in \Re_{++}^{L_2-1} : \exists \, \mathbf{l}_p \; s.t. \; \pi_2 \in \hat{E}_2(\hat{\mathbf{x}}_1, \pi_1, q, \mathbf{l}_p), \\ for \; some \; (\hat{\mathbf{x}}_1, \pi_1, q) \in \hat{E}_1(\mathbf{l}_p) \end{array} \right\} \cap \Pi_{2,p}^{n-1}$$

Obviously, $\Pi_{2,p}^n \subseteq \Pi_{2,p}^{n-1}$, $n = 1, ...$

Rationalizable (second period) price expectations: $\tilde{\Pi}_{2,p} = \Pi_{2,p}^\infty$.

Partial Consensus outcomes: For $0 \leq p < 1$, given $\Pi_2^0 \subset \Re_{++}^{L_2-1}$, a consensus outcome is a triple $(\tilde{\pi}_1, \tilde{q}, \tilde{\pi}_2)$ and an allocation $(\tilde{\mathbf{x}}_1, \tilde{\mathbf{y}}, \tilde{\mathbf{x}}_2)$ such that $(\tilde{\mathbf{x}}_1, \tilde{\pi}_1, \tilde{q}) \in \hat{E}_1(\mathbf{l}_p)$ for some $\mathbf{l}_p$ on $\tilde{\Pi}_{2,p}$ with probability at least p on $\tilde{\pi}_2 \in \tilde{\Pi}_{2,p}$.

Proposition 6. Consider a PFE vector of prices $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$. Suppose $u^i(x_1, x_2) = u_1^i(x_2) + u_2(x_2)$ where $u_2(.)$ homothetic for all $i \in I$. Then, for any $\Pi_2^0 \subset \Re_{++}^{L_2-1}$,such that $\hat{\pi}_2 \in \Pi_2^0$, $\tilde{\Pi}_{2,p} = \{\tilde{\pi}_2\}$ for all $0 \leq p < 1$.

Proof. As market clearing in both periods is common knowledge $\int \hat{\mathbf{y}}^i di = 0$, and therefore, $\int d\hat{\mathbf{y}}^i di = 0$. As $u_2^i(x_2)$ is identical and homothetic for all $i \in I$,

$$\partial_y \hat{\mathbf{x}}_2^i(\hat{\pi}_2, \hat{\mathbf{y}}^i(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)) = \partial_y \hat{\mathbf{x}}_2^j(\hat{\pi}_2, \hat{\mathbf{y}}^j(\hat{\pi}_1, \hat{q}, \hat{\pi}_2))$$

for all $i, j \in I$, and therefore, $\int \partial_y \hat{\mathbf{x}}_2^i d\hat{\mathbf{y}}^i di = \partial_y \hat{\mathbf{x}}_2^i \int d\hat{\mathbf{y}}^i di = 0$ so that as long as $\hat{\pi}_2 \in \Pi_2^0$, $\hat{E}_2(\mathbf{x}_1, \pi_1, q, \mathbf{l}_p) = \{\hat{\pi}_2\}$ for all $(\pi_1, q) \in \hat{E}_1(\mathbf{l}_p)$, $0 \leq p < 1$. Therefore, $\tilde{\Pi}_{2,p} = \{\hat{\pi}_2\}$.■

A partial consensus outcomes reduces to a PFE when all individuals are able to continue eliminating prices till $\tilde{\pi}_2$ is the only element in $\tilde{\Pi}_{2,p}$. When preferences are additively separable in consumption across the two time periods, a change in the asset holdings in period 1 amounts to a redistribution of revenue in period 2 spot markets. When preferences over consumption in period 2 spot markets are identical and homothetic, a redistribution of revenue will have no impact on period 2 spot prices. Therefore, as long the PFE second period price $\hat{\pi}_2$ is in Π_2^0, it is the only price vector consistent with market clearing in period 2 irrespective of what second price expectations individuals started out with: it is the *unique* rationalizable second price expectation.

We are now in a position to examine the $p-$stability of a PFE:

Proposition 7. Consider a sequentially regular PFE vector of prices $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$. Whenever the degree of consensus p on $\hat{\pi}_2 \in \Pi_2^0 \subseteq N(\hat{\pi}_2, \varepsilon)$ of $\hat{\pi}_2$ is higher than a critical threshold value $\overline{p} < 1$, the PFE is the unique rationalizable outcome.

Proof. When the PFE is sequentially regular, then the admissibility condition required for Proposition 3 is satisfied. By relabelling variables appropriately, the result is an immediate consequence of Proposition 3. ■

Fix a strongly regular $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$. Clearly, as long as $\hat{\pi}_2 \in \Pi_2^0 \subseteq N(\hat{\pi}_2, \varepsilon)$ of $\hat{\pi}_2$, there is a partial consensus outcome that is Pareto optimal. However, by

Proposition 1, when $\tilde{\Pi}_{2,p}$ has a non-empty interior in $\Re^{L_2-1}_{++}$, it also contains partial consensus outcomes are distinct from a PFE such that the associated allocations aren't Pareto optimal. Evidently, the marginal rates of substitution will be equalized across individuals in spot markets within a time period but not across time periods. Moreover, at a strongly regular $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$, the associated partial consensus asset price $\tilde{q} \neq \hat{q}$: hence, there is an asset price bubble.

In what follows, starting from a fixed strongly regular $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$ and a Π^0_2 such that $\hat{\pi}_2 \in \Pi^0_2$, a local characterization of the as set of sufficient conditions that ensures the existence of a $\tilde{\Pi}_{2,p}$ that has a non-empty interior in $\Re^{L_2-1}_{++}$ is carried out for p small enough. Suppose $\Pi^0_2 = N(\hat{\pi}_2, \varepsilon), \Pi^0_2 \neq \{\hat{\pi}_2\}$ with $N(\hat{\pi}_2, \varepsilon) \subset \Re^{L_2-1}_{++}$ a neighborhood around $\hat{\pi}_2$, the second period component of a PFE vector of prices $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$. Let $\|.\|$ be a monotone vector norm[9] on $\Re^{L_2-1}$. Let $S(\varepsilon) = \{z \in \Re^{L_2-1}_{++} : \|z - \hat{\pi}_2\| = \varepsilon, \varepsilon > 0\}$. Let $B(\bar{\varepsilon}) = \{x \in \Re^{L_2-1}_{++} : \|x - \hat{\pi}_2\| < \bar{\varepsilon}\}$.

As the economy is sequentially regular, it will be convenient to provide a local characterization of the map from each assignment of expectations $\mathbf{f} : I \to \Pi^0_2$ (remember $p = 0$) to a market clearing price in the second period π_2 in the vicinity of a PFE:

Lemma 2: Fix a sequentially regular PFE $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$. There exists a neighborhood $N(\hat{\pi}_2, \varepsilon)$ of $\hat{\pi}_2$ and matrices M^i for each $i \in I$ such that for each assignment of expectations $\mathbf{f} : I \to \Pi^0_2$, $\Pi^0_2 = N(\hat{\pi}_2, \varepsilon)$, $d\pi_2 = \int \mathbf{M}^i d\mathbf{f}^i di$ where $d\pi_2 = (\pi_2 - \hat{\pi}_2)$ and $d\mathbf{f}^i = (\mathbf{f}^i - \hat{\pi}_2)$. Moreover $\tilde{\Pi}_{2,0} = \{\pi_2 \in N(\hat{\pi}_2, \varepsilon) : d\pi_2 = \int \mathbf{M}^i d\mathbf{f}^i di,\ for\ some\ \ \mathbf{f} : I \to \tilde{\Pi}_{2,0}\}$.

Proof. See appendix.■

What conditions ensure that redistributions of revenue in second period markets or when redistributions of commodities in period 1 change second period spot market prices? The following proposition provides an answer:

Proposition 8. (a) If there exists $\bar{\varepsilon} > 0$ such that $\Pi^0_2 \subseteq N(\hat{\pi}_2, \bar{\varepsilon})$, (a)$\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\| < \bar{\varepsilon}$, for all assignment of expectations $\mathbf{v} : I \to S(\bar{\varepsilon})$, then $\tilde{\Pi}_{2,p} = \{\hat{\pi}_2\}$, (b) if $\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\| > \varepsilon$, for all assignment of expectations $\mathbf{v} : I \to S(\varepsilon)$, for each $\varepsilon \leq \bar{\varepsilon}$, $N(\hat{\pi}_2, \bar{\varepsilon}) \subseteq \tilde{\Pi}_{2,p}$ for $p < \tilde{p}$, for some $\tilde{p} > 0$. Moreover, the condition that $\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\| < \bar{\varepsilon}$ is invariant to the choice of the second period numeraire. (b) Consider a sequentially regular PFE vector of prices $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$. If there exists $\tilde{\varepsilon} > 0$ such that either (i)

$$\max\left\{\left\|\partial_y \hat{\mathbf{x}}^i_2(\hat{\mathbf{x}}^i_1, \hat{\pi}_2, \hat{\mathbf{y}}^i) - \partial_y \hat{\mathbf{x}}^j_2(\hat{\mathbf{x}}^i_1, \hat{\pi}_2, \hat{\mathbf{y}}^i)\right\|, \left\|\partial_{x_1} \hat{\mathbf{x}}^i_2(\hat{\mathbf{x}}^i_1, \hat{\pi}_2, \hat{\mathbf{y}}^i) - \partial_{x_1} \hat{\mathbf{x}}^j_2(\hat{\mathbf{x}}^i_1, \hat{\pi}_2, \hat{\mathbf{y}}^i)\right\|\right\} < \tilde{\varepsilon}$$

for all $i, j \in I$, or (ii) $\max\left\{\left\|\partial_{\pi_2} \hat{\mathbf{y}}^i\right\|, \left\|\partial_{\pi_2} \hat{\mathbf{x}}^i_1\right\|\right\} < \tilde{\varepsilon}$ for all $i \in I$, there exists a neighborhood $N(\hat{\pi}_2, \varepsilon)$ of $\hat{\pi}_2$ and $\Pi^0_2 \subset N(\hat{\pi}_2, \varepsilon)$ such that $\tilde{\Pi}_{2,p} = \{\hat{\pi}_2\}$, $0 \leq p \leq 1$.

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[9] For any $x \in \Re^K$, let $|x| = (|x_1|, ..., |x_K|)$. It follows that $|x| \geq |y|$ if and only if $|x_l| \geq |y_l|$ for all $l = 1, ..., K$. A vector norm, $\|.\|$ on $\Re^K$ is monotone if and only if $|x| \geq |y| \implies \|x\| \geq \|y\|$. All l_p norms, including the euclidean norm, are monotone. However, (see Horn and Johnson (1985)) the following vector norm on $\Re^K$, $\|x\| = |x_1 - x_2| + \sum_{l' \neq 1} |x_{l'}|$, is not monotone.

Proof. See appendix.■

Heuristically, when (a) second period spot market prices are sensitive to redistributions of revenue in second period markets or a redistribution of commodities in period 1 change second period spot market prices, *and* (b) there is lack of consensus over second period prices (so that beliefs over second period prices are heterogeneous to a sufficient degree), then an asset pice bubble exists. Conversely, even with lack of consensus over future prices in a small enough neighborhood of a perfect foresight equilibria, an asset price bubble will not exist if (a) the sensitivity of second period spot commodity prices to a redistribution of revenue in the same period and a redistribution in period 1 endowments is small and (b) the sensitivity of period 1 consumption and asset demands to small changes in expectations of second period prices is small.

Notice that when the conditions under which proposition 8a holds, any asset market price in an appropriately chosen interval of the PFE asset market price can be a partial consensus asset market price. Therefore, local bubbles exist.

4.3 A global analysis of bubbles and preference heterogeneity: an example

In this part of the paper, we develop a simple example of intertemporal trade with an unique PFE with no trade in asset markets. In a global analysis, we demonstrate how the interaction of heterogeneous beliefs and heterogeneous preferences implies the existence of $p-$consensus outcomes characterized by asset price bubbles. The distance of a PFE asset price a $p-$consensus asset price is entirely constrained by the heterogeneity of preferences.

There is a continuum $[0,2]$ of infinitesimal agents, 2 groups $i = A, B$ (size 1 each) of agents. 2 periods $t = 1, 2$. At $t = 1$, there is a single commodity available for consumption and as before agents can transfer revenue over time using an asset in zero net supply. There are two commodities available for consumption at $t = 2$ and as before, asset payoffs are denoted in commodity, the numéraire good. Denote π_2 the relative of price of commodity 2 at $t = 2$.

Every agent in group A has an intertemporal utility function $u^A(x_{1,1}, x_{1,2}, x_{2,2}) = \frac{(x_{1,1})^{1-\sigma}}{1-\sigma} + (x_{1,2})^{\alpha}(x_{2,2})^{1-\alpha}$, $\sigma \geq 0$, $\sigma \neq 1$, $0 < \alpha < 1$; every agent in group B has an intertemporal utility function $u^B(x_{1,1}, x_{1,2}, x_{2,2}) = \frac{(x_{1,1})^{1-\sigma}}{1-\sigma} + (x_{1,1})^{1-\alpha}(x_{2,2})^{\alpha}$. We will assume $\alpha > 1/2$. Note α is a measure of preference heterogeneity between the two groups of agents: closer α is to one, the higher the degree of preference of preference heterogeneity between the two groups.

To focus on the role of preference heterogeneity in generating $p-$consensus outcomes, we will assume that both groups of agents have the same endowments i.e. for $i = A, B$, $\left(\mathbf{w}^i_{1,1}, \mathbf{w}^i_{1,2}, \mathbf{w}^i_{2,2}\right) = (\mathbf{w}_1, \mathbf{w}_{1,2}, \mathbf{w}_{2,2})$. Furthermore, we will assume that $\mathbf{w}_{1,2} = \mathbf{w}_{2,2} = \mathbf{w}_2$. For simplicity, we also will assume that all the agents in group i have the same asset holding, denoted y^i i.e. all agents in a group share the same price expectations and have therefore the same optimal behavior.

The budget constraint at $t = 0$ for all agents is $x_{1,1} + qy^i = \mathbf{w}_1$, $i = A, B$.

As already noted, we want to avoid bankruptcy in period 2. For $i = A, B$, when $y^i < 0$, an agent borrows at $t = 1$ with a promise to repay at $t = 2$ we may have $\mathbf{w}_2(1+\pi_2)+y^i < 0$ (i.e. the agent defaults on her debt). Given π_2, y^i, we define the $t = 2$ wealth $W^i(\pi_2, y^i)$ of an agent of group $i = A, B$ as follows: if $y^i < 0$, is $W^i(\pi_2, y^i) = \max\left\{0, \mathbf{w}_2(1+\pi_2)+y^i\right\}$; otherwise $W^i(\pi_2, y^i) = \mathbf{w}_2(1+\pi_2)y'^i$ where $0 \leq y'^i \leq y^i$ (an agent may not get back all that is owed to her).

Therefore, at $t = 2$, three cases are possible: if $W^i(\pi_2, y^i) \geq 0$ for both $i = A, B$, the wealth of an agent in group $i = A, B$ is $\mathbf{w}_2(1+\pi_2)+y^i$; $W^i(\pi_2, y^i) < 0$ (group i defaults), asset market clearing $y^i+y^j = 0$ at $t = 1$ implies $W^j(\pi_2, y^j) > 0$ so that $W^i(\pi_2, y^i) = 0$ and $W^j(\pi_2, y^j) = 2\mathbf{w}_2(1+\pi_2)$, $j \neq i$, $i, j = A, B$, (i.e. each the defaulting agent gives everything to some market maker who then redistributes the wealth to all creditors equally),

At the beginning of $t = 1$, the price π_2 depends on asset holdings. As preferences over period 2 commodities are homothetic preferences, period 2 demand (and indirect utility) is linear in wealth. Market clearing for good 1 is $\alpha W^A(\pi_2, y^A) + (1-\alpha) W^B(\pi_2, y^A) = 2\mathbf{w}_2$ so that when $W^i(\pi_2, y^i) \geq 0$ for both $i = A, B$ (there is no default), the market clearing relative price is $\pi_2 = 1 - \frac{(2\alpha-1)y^A}{\mathbf{w}_2}$ and we must have that $-y_B \leq \frac{\mathbf{w}_2}{\alpha}$ as well as $-y_A \leq \frac{\mathbf{w}_2}{(1-\alpha)}$ (i.e. debt is bounded). When agents in group A default, $\pi_2^{\max} = \frac{\alpha}{1-\alpha}$, and $-y_A > \frac{\mathbf{w}_2}{(1-\alpha)}$; when agents in group B default, $\pi_2^{\min} = \frac{1-\alpha}{\alpha}$, and $-y_B > \frac{\mathbf{w}_2}{\alpha}$.

Hence, allowing for asset market clearing in period 1, at $t = 2$, we must have that $\pi_2 \in \Pi_2^0 = \left[\pi_2^{\min}, \pi_2^{\max}\right] = \left[\frac{1-\alpha}{\alpha}, \frac{\alpha}{1-\alpha}\right]$. Notice that set of period 2 prices compatible with no default is entirely determined by the parameter α: the closer α is to one (the more heterogenous preferences are), the bigger is the set Π_2^0.

The indirect utility over period two consumption for each group is given as follows: $v^A\left(\pi_2, W^A(\pi_2, y^A)\right) = \frac{\alpha^\alpha(1-\alpha)^{1-\alpha}}{\pi_2^{1-\alpha}} W^A(\pi_2, y^A)$ and $v^B\left(\pi_2, W^B(\pi_2, y^B)\right) = \frac{\alpha^\alpha(1-\alpha)^{1-\alpha}}{\pi_2^\alpha} W^B(\pi_2, y^B)$. Evidently, these are linear in W^i and hence, piecewise linear in y^i, $i = A, B$.

Let $y_A^{\min} = -\frac{\mathbf{w}_1}{(1-\alpha)}$ and $y_A^{\max} = \frac{\mathbf{w}_1}{\alpha}$. Then, no default in period 2 implies (using ,asset market clearing in period one so that $y_B = -y_A$) implies the following: $y^A \in \left[y_{\min}^A, y_{\max}^A\right]$ and $y^B \in \left[y_{\min}^B, y_{\max}^B\right] = \left[-y_{\max}^A, -y_{\min}^A,\right]$.

Hence at $t = 0$, a group A individual maximizes

$$\max_{y^A \in \left[y_{\min}^A, y_{\max}^A\right]} \frac{\left(\mathbf{w}_1 - qy^A\right)^{1-\sigma}}{1-\sigma} + \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{\pi_2^{1-\alpha}} W^A(\pi_2, y^A)$$

and a group B individual maximizes

$$\max_{y^B \in \left[y_{\min}^B, y_{\max}^B\right]} \frac{\left(\mathbf{w}_1 - qy^B\right)^{1-\sigma}}{1-\sigma} + \frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{\pi_2^\alpha} W^B(\pi_2, y^B)$$

A straightforward computation show that at a PFE $\widehat{\pi}_2 = 1$, $\widehat{q} = (\mathbf{w}_1)^\sigma \alpha^\alpha (1-\alpha)^{1-\alpha}$and

$\widehat{y}^A = \widehat{y}^B = 0$. Observe that at the unique PFE, there is no trade in asset markets.

To characterize the set of $p-$consensus outcomes, to begin with, we assume that $\sigma = 0$. In this case, observe that intertemporal utility function is piece wise linear in the amount of the asset held. Given a p-belief over $\pi_2 \in \Pi_2^0$, let $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)$ (respectively, $E_p\left(\frac{1}{\pi_2^{\alpha}}\right)$) denote the expected value of π_2 for a group A agent (resp., group B agent). For a given q and a p-belief over $\pi_2 \in \Pi_2^0$, the optimal y^A is determined as follows:

$$y^A\left(q, E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right) = \begin{cases} y^A_{\min} \text{ if } q > \alpha^\alpha (1-\alpha)^{1-\alpha} E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) \\ \left[y^A_{\min}, y^A_{\max}\right] \text{ if } q = \alpha^\alpha (1-\alpha)^{1-\alpha} E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) \\ y^A_{\max} \text{ if } q < \alpha^\alpha (1-\alpha)^{1-\alpha} E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) \end{cases}$$

The optimal $y^B\left(q, E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right)$ is defined analogously.

No asset price q can be strictly smaller (resp., strictly greater) than $\alpha^\alpha (1-\alpha)^{1-\alpha} E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)$ and $\alpha^\alpha (1-\alpha)^{1-\alpha} E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)$ clears the market, as $y^A\left(q, E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right) > 0$ and $y^B\left(q, E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right) > 0$ (resp., as both $y^A\left(q, E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right) < 0$ and $y^B\left(q, E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right) < 0$). Hence, the asset market clearing price is determined as follows:

$$q'\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right)$$
$$= \begin{cases} \alpha^\alpha (1-\alpha)^{1-\alpha} E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), \text{ if } E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) = E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right) \\ \left[E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right], \text{ if } E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) < E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right) \\ \left[E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right), E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right], \text{ if } E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) > E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right) \end{cases}$$

The market clearing π_2' price at $t = 2$ is determined as follows: if $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) < E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)$, $y^A\left(q, E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right) < 0$ and $y^B\left(q, E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right) > 0$ so that $\pi_2' = \pi_2^{\max}$; if $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) > E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)$, $y^A\left(q, E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right) > 0$ and $y^B\left(q, E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right) < 0$, so that $\pi_2' = \pi_2^{\min}$; if $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) = E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)$, $y^A\left(q, E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right) \in \left[y^A_{\min}, y^A_{\max}\right]$ and $\pi_2' \in [\pi_2^{\min}, \pi_2^{\max}]$.

At an interior PFE $\widehat{\pi}_2 \in (\pi_2^{\min}, \pi_2^{\max})$, we must have that $\frac{1}{\widehat{\pi}_2^{1-\alpha}} = \frac{1}{\widehat{\pi}_2^{\alpha}}$. As $1 \in (\frac{1-\alpha}{\alpha}, \frac{\alpha}{1-\alpha}) = (\pi_2^{\min}, \pi_2^{\max})$, the unique PFE is $\widehat{\pi}_2 = 1$ with $y^A = y^B = 0$. Note, however, when $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) = E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right) = 1$, any $y^A \in \left[y^A_{\min}, y^A_{\max}\right]$ is an optimal choice. So, the optimal choice of an agent isn't uniquely defined at a PFE. Hence, the PFE is $1-$stable but can never be $p-$stable for $p < 1$.

It remains to compute the partial consensus outcomes. Given a $p-$belief on a price $\pi_2 \in \Pi_2^0$, we must have that

$$E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) \in \left[\frac{p}{\pi_2^{1-\alpha}} + \frac{1-p}{\left(\pi_2^{\max}\right)^{1-\alpha}}, \frac{p}{\pi_2^{1-\alpha}} + \frac{1-p}{\left(\pi_2^{\min}\right)^{1-\alpha}}\right]$$
$$E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right) \in \left[\frac{p}{\pi_2^{\alpha}} + \frac{1-p}{\left(\pi_2^{\max}\right)^{\alpha}}, \frac{p}{\pi_2^{\alpha}} + \frac{1-p}{\left(\pi_2^{\min}\right)^{\alpha}}\right]$$

Hence, an interior price $\pi_2 \in (\pi_2^{\min}, \pi_2^{\max})$ is a partial consensus outcome if and only if the both the preceding intervals intersect.

If $\pi_2 > 1$, we require that $\frac{p}{\pi_2^{\alpha}} + \frac{1-p}{\left(\pi_2^{\min}\right)^{\alpha}} \geq \frac{p}{\pi_2^{1-\alpha}} + \frac{1-p}{\left(\pi_2^{\max}\right)^{1-\alpha}}$ since $\frac{1}{\pi_2^{\alpha}} < \frac{1}{\pi_2^{1-\alpha}}$ (because $\alpha > 1/2$). The intuition is that agents in A must be pessimistic enough (so that $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)$ is less than $\frac{1}{\pi_2^{1-\alpha}}$) and agents in B must be optimistic enough (so that $E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)$ is greater than $\frac{1}{\pi_2^{\alpha}}$), so this cannot be possible if p is too close to 1. Note that

$$\frac{p}{\pi_2^{\alpha}}+\frac{1-p}{\left(\pi_2^{\min}\right)^{\alpha}} \geq \frac{p}{\pi_2^{1-\alpha}}+\frac{1-p}{\left(\pi_2^{\max}\right)^{1-\alpha}} \Leftrightarrow p \leq \frac{\frac{1}{\left(\pi_2^{\min}\right)^{\alpha}} - \frac{1}{\left(\pi_2^{\max}\right)^{1-\alpha}}}{\frac{1}{\left(\pi_2^{\min}\right)^{\alpha}} - \frac{1}{\left(\pi_2^{\max}\right)^{1-\alpha}} + \frac{1}{\pi_2^{1-\alpha}} - \frac{1}{\pi_2^{\alpha}}} =^{def} p^c\left(\pi_2\right)$$

where $\frac{1}{\pi_2^{1-\alpha}} - \frac{1}{\pi_2^{\alpha}} \geq 0$ (as $\alpha > \frac{1}{2}$) and $\frac{1}{\left(\pi_2^{\min}\right)^{\alpha}} - \frac{1}{\left(\pi_2^{\max}\right)^{1-\alpha}} > 0$ as $\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2\alpha-1}} > 1$ (again, as $\alpha > \frac{1}{2}$). Moreover, $p^c\left(\pi_2\right)$ tends to 1 when π_2 tends to 1 (as $\frac{1}{\pi_2^{1-\alpha}} - \frac{1}{\pi_2^{\alpha}}$ tends to 0).

If $\pi_2 < 1$, we require that $\frac{p}{\pi_2^{\alpha}} + \frac{1-p}{\left(\pi_2^{\max}\right)^{\alpha}} \leq \frac{p}{\pi_2^{1-\alpha}} + \frac{1-p}{\left(\pi_2^{\min}\right)^{1-\alpha}}$ since $\frac{1}{\pi_2^{\alpha}} > \frac{1}{\pi_2^{1-\alpha}}$ so that, by an analogous computation to the one above, we obtain that

$$\frac{p}{\pi_2^{\alpha}}+\frac{1-p}{\left(\pi_2^{\max}\right)^{\alpha}} \leq \frac{p}{\pi_2^{1-\alpha}}+\frac{1-p}{\left(\pi_2^{\min}\right)^{1-\alpha}} \Leftrightarrow p \leq \frac{\frac{1}{\left(\pi_2^{\min}\right)^{1-\alpha}} - \frac{1}{\left(\pi_2^{\max}\right)^{\alpha}}}{\frac{1}{\left(\pi_2^{\min}\right)^{1-\alpha}} - \frac{1}{\left(\pi_2^{\max}\right)^{\alpha}} + \frac{1}{\pi_2^{\alpha}} - \frac{1}{\pi_2^{1-\alpha}}} =^{def} p^c\left(\pi_2\right)$$

As before, $p^c\left(\pi_2\right)$ tends to 1 when π_2 tends to 1 (as $\frac{1}{\pi_2^{1-\alpha}} - \frac{1}{\pi_2^{\alpha}}$ tends to 0). Therefore, the level of $p-$consensus is arbitrarily high for prices π_2 close to the interior PFE with an upper bound π_2 is from one.

Note that at a PFE, $\widehat{q} = \alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}$; at a partial consensus outcome with $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) = E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right) \neq 1$, $q' = \alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha} E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) \neq \alpha^{\alpha}\left(1-\alpha\right)^{1-\alpha}$. Hence, a partial consensus asset price can be made large or small relative to the unique PFE by choosing different values of $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)$ constrained entirely by the heterogeneity of the preferences.

One concern with the analysis so far is that by focusing on the case with $\sigma = 0$, the characterization of $p-$consensus outcomes may not be robust. We address this concern by carrying out an analysis of $p-$consensus outcomes for values of σ in the vicinity of zero. When $\sigma \neq 0$, the first order conditions characterizing an interior optimum to the intertemporal optimization problem for a group A individual is:

$$q\left(\mathbf{w}_1 - qy^A\right)^{-\sigma} = \alpha^\alpha (1-\alpha)^{1-\alpha} E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)$$

so that

$$y^A\left(q, E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right) = \frac{1}{q}\left[\mathbf{w}_1 - \left(\frac{q}{\alpha^\alpha (1-\alpha)^{1-\alpha}}\right)^{\frac{1}{\sigma}} \left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right)^{-\frac{1}{\sigma}}\right]$$

An symmetric computation yields that

$$y^B\left(q, E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right) = \frac{1}{q}\left[\mathbf{w}_1 - \left(\frac{q}{\alpha^\alpha (1-\alpha)^{1-\alpha}}\right)^{\frac{1}{\sigma}} \left(E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right)^{-\frac{1}{\sigma}}\right]$$

Asset market clearing requires that $y^A + y^B = 0$. Hence, by computation, the market clearing asset price is

$$\begin{aligned} & q'\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right) \\ & = (2\mathbf{w}_1)^\sigma \alpha^\alpha (1-\alpha)^{1-\alpha} \left[\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right)^{-\frac{1}{\sigma}} + \left(E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right)^{-\frac{1}{\sigma}}\right]^{-\sigma} \end{aligned}$$

The market clearing second period commodity price is

$$\begin{aligned} & \pi_2'\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right) \\ & = 1 - \frac{(2\alpha-1)}{\mathbf{w}_2 q'\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right)} \left[\begin{array}{c} \mathbf{w}_1 - \\ \left(\frac{q'\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right)}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)^{\frac{1}{\sigma}} \left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right)\right)^{-\frac{1}{\sigma}} \end{array}\right] \end{aligned}$$

Evaluated at the PFE, by computation, it is checked that

$$\left|\frac{\partial \pi_2'(1,1)}{\partial \pi_2}\right| = (2\alpha-1)(1-p)\left|\left[\begin{array}{c} -2^{2\sigma-1}\mathbf{w}_1^\sigma \alpha^\alpha (1-\alpha)^{1-\alpha}\left(\frac{1-\alpha}{\pi_2^{2-3\alpha}} + \frac{\alpha}{\pi_2^{1-\alpha}}\right) \\ +\frac{2^{-2\sigma}\mathbf{w}_1^{1-\sigma}}{\sigma}\left(\frac{(1+\sigma)}{2}\left(\frac{1-\alpha}{\pi_2^{2-3\alpha}} + \frac{\alpha}{\pi_2^{1-\alpha}}\right) - \frac{(1-\alpha)}{\pi_2^{2-3\alpha}}\right) \end{array}\right]\right|$$

Hence, the closer p is to 1, the flatter is the derivative $\left|\frac{\partial \pi_2'(1,1)}{\partial \pi_2}\right|$. Therefore, for each value of $\sigma > 0$, the PFE is $p-$stable. Moreover, for a fixed value of $p < 1$, $p \geq 0$, $\lim_{\sigma \longrightarrow 0}\left|\frac{\partial \pi_2'(1,1)}{\partial \pi_2}\right| = \infty$. Hence, in the limit, as

$\sigma \to 0$, the PFE is $1-$stable, consistent with the computation for the case with $\sigma = 0$ presented above. Furthermore, by computation, for $\pi_2 \in (\pi_2^{\min}, \pi_2^{\max})$, $\lim_{\sigma \longrightarrow 0} \left| \frac{\partial \pi_2'\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right)}{\partial \pi_2} \right| = \infty$. Therefore, there exists $\overline{\sigma} > 0$ such that for all $\sigma \in [0, \overline{\sigma})$, $\left| \frac{\partial \pi_2'\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right)}{\partial \pi_2} \right| > 1$ for all $\pi_2 \in (\pi_2^{\min}, \pi_2^{\max})$. Hence, each $\pi_2 \in (\pi_2^{\min}, \pi_2^{\max})$ is rationalizable and therefore, by Proposition 2, a $p-$consensus outcome.

At a partial consensus outcome asset prices are $q'\left(E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right), E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right)\right) \neq \widehat{q}$. Hence, as before, a partial consensus asset price can be made large or small relative to the unique PFE by choosing different values of $E_p^A\left(\frac{1}{\pi_2^{1-\alpha}}\right) \neq 1$ and $E_p^B\left(\frac{1}{\pi_2^{\alpha}}\right) \neq 1$ constrained entirely by the heterogeneity of the preferences.

5 Conclusion

In this paper, we have developed a new solution concept that allows for partial consensus about the outcomes of strategic and market interaction and an associated, continuous measure of the degree of stability, via belief coordination, for equilibrium outcomes. In a number of key results and examples, we have illustrated the properties of our concepts. We have examined the foundations of intertemporal trade via belief coordination in a two period economy and show that, under certain conditions, lack of consensus over future prices is consistent with an asset price bubble.

Our contributions are a preliminary step towards understanding how the possibility that non-equilibrium outcomes are approximately self-fulfilling complicate the analysis of strategic interaction and market behavior. In future research, we intend to examine this point in grater detail to obtain new insights in a variety of economic applications and their implications for policy.

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Appendix

Proof of Lemma 1.

(i) For every M, for every $\varepsilon > d_{\Delta(A)}\left(\tau_a', \widehat{\tau}_a\right)$, we have

$$\widehat{\tau}_a\left(M\right) \leq \tau_a'\left(M^{\varepsilon}\right) + \varepsilon,$$

then

$$\begin{aligned}(1-p)\,\widehat{\tau}_a\left(M\right) &\leq (1-p)\,\tau_a'\left(M^{\varepsilon}\right) + (1-p)\,\varepsilon, \\ \widehat{\tau}_a\left(M\right) &\leq p\widehat{\tau}_a\left(M\right) + (1-p)\,\tau_a'\left(M^{\varepsilon}\right) + (1-p)\,\varepsilon, \\ \widehat{\tau}_a\left(M\right) &\leq p\widehat{\tau}_a\left(M^{\varepsilon}\right) + (1-p)\,\tau_a'\left(M^{\varepsilon}\right) + (1-p)\,\varepsilon.\end{aligned}$$

This implies: $d_{\Delta(A)}(\tau_a, \widehat{\tau}_a) \leq (1-p)\,\varepsilon$ and $d_{\Delta(A)}(\tau_a, \widehat{\tau}_a) \leq (1-p)\, d_{\Delta(A)}(\tau'_a, \widehat{\tau}_a)$.

(ii) For a Borel set M s.t. $x \in M$, for every ε, we have $\delta_x(M^\varepsilon) = 1$ and

$$\tau_a(M) \leq \delta_x(M^\varepsilon) + \varepsilon.$$

For a Borel set M that does not intersect S, for every ε, we have $\tau_a(M) = 0$ and

$$\tau_a(M) \leq \delta_x(M^\varepsilon) + \varepsilon.$$

Consider now a Borel set M that does not contain x and that intersects S. For every $\varepsilon > d$, we have that $x \in M^\varepsilon$ (consider a y in $S \cap M$) and $\delta_x(M^\varepsilon) = 1$ and

$$\tau_a(M) \leq \delta_x(M^\varepsilon) + \varepsilon.$$

(iii) Consider $\varepsilon > \sup_\lambda \mathtt{ess}\ d_{\Delta(A)}(\tau_\lambda, \nu)$. For $f-$almost every λ, we have: for every M

$$\tau_\lambda(M) \leq \nu(M^\varepsilon) + \varepsilon.$$

Summing over λ gives:

$$\int \tau_\lambda(M)\, f(d\lambda) \leq \int (\nu(M^\varepsilon) + \varepsilon)\, f(d\lambda) = \nu(M^\varepsilon) + \varepsilon,$$

as $(\nu(M^\varepsilon) + \varepsilon)$ does not depend on λ and $\int f(d\lambda) = 1$. ■

Proof of Proposition 3.

There is a neighborhood $N \subset \Delta(A)$ of τ_a^* such that for $\mu-$almost every u in U_A,

$$\forall m \in N, d_A(B(u,m), B(u, \tau_a^*)) \leq K d_{\Delta(A)}(m, \tau_a^*). \quad (2)$$

Now for each $\tau \in T_p$ (with $\tau = p\tau^* + (1-p)\tau'$), we must have that

$$\forall M \subset A, \tau_a(M) = p\tau_a^*(M) + (1-p)\,\tau'_a(M),$$

It straightforwardly follows from Lemma 1(i) that

$$d_{\Delta(A)}(\tau_a, \tau_a^*) \leq (1-p)\, d_{\Delta(A)}(\tau'_a, \tau_a^*). \quad (3)$$

As the Prohorov metric is always bounded by 1, we have $d_{\Delta(A)}(\tau'_a, \tau_a^*) \leq 1$ and $d_{\Delta(A)}(\tau_a, \tau_a^*) \leq 1-p$. Then, for p large enough, the following property holds: the inequality (2) holds for the marginal τ_a of any distribution τ in T_p. From now on, we consider p such that this property holds. Define the set $A^n(u) \subset A$ of actions that are best responses of u to a distribution of actions τ_a that is the marginal on A of some $\tau \in S_p^{n-1}$. $\phi\left(S_p^{n-1}\right)$ contains the distributions $\tau \in T$ such that, for μ-almost every u, $\tau(A^n(u)\,|u) = 1$. For μ-almost every u, for every a in $A^n(u)$, a writes $B(u, \tau_a)$ for some τ in S_p^{n-1}. Inequality (2) writes:

$$d_A(a, B(u, \tau_a^*)) \leq K d_{\Delta(A)}(\tau_a, \tau_a^*).$$

As $\tau \in S_p^{n-1} = \phi\left(S_p^{n-2}\right) \cap T_p$, $\tau = p\tau^* + (1-p)\,\tau'$ for some $\tau' \in \phi\left(S_p^{n-2}\right)$. Inequality (3) implies

$$d_A(a, B(u, \tau_a^*)) \leq K\,(1-p)\, d_{\Delta(A)}(\tau'_a, \tau_a^*).$$

Denote $R_A^n(u) = \sup_{a \in A^n(u)} d_A(a, B(u, \tau_a^*))$ (this is the radius of the smallest ball containing $A^n(u)$ and centered on τ_A^*). We have

$$R_A^n(u) \leq K(1-p)\, d_{\Delta(A)}(\tau_a', \tau_a^*).$$

Denote $R_A^n = \sup \mathtt{ess}_{u \in U_A}\ R_A^n(u)$ for every n. We have

$$R_A^n \leq K(1-p)\, d_{\Delta(A)}(\tau_a', \tau_a^*). \tag{4}$$

Consider now that by definition, for every M, $\tau_a^*(M) = \int \tau^*(M|u)\, \mu(du)$. By admissibility of τ^*, for $\mu-$almost every u, the conditional distribution $\tau^*(.|u)$ is a Dirac measure $\delta_{B(\tau_a^*,u)}$ on the equilibrium action of u (denoted $B(\tau_a^*, u)$). By Lemma 1(ii), for every Dirac measure centered on $x \in A$,

$$d_{\Delta(A)}(\tau_a', \delta_x) \leq \sup_{y \in S} d_A(x, y), \tag{5}$$

where S is the support of τ_a' (the smallest closed set such that $\tau_a'(S) = 1$). Then, we have:

$$d_{\Delta(A)}(\tau_a', \delta_{B(\tau_a^*,u)}) \leq R_A^{n-2}(u)$$

As $\tau_a^* = \int \delta_{B(\tau_a^*,u)} \mu(du)$, by Lemma 1(iii),

$$d_{\Delta(A)}(\tau_a', \tau_a^*) \leq \sup_{u \in U_A} \mathtt{ess}\ d_{\Delta(A)}(\tau_a', \delta_{B(\tau_a^*,u)}) \leq \sup_{u \in U_A} \mathtt{ess}\ R_A^{n-2}(u) = R_A^{n-2}.$$

From Inequality (4), we have

$$R_A^n \leq K(1-p)\, R_A^{n-2}.$$

Hence, for p large enough, $K(1-p) < 1$ and the sequence of R_A^n tends to 0, which implies that S_p^n tends to $\{\tau^*\}$. We have shown $p-$stability for $p < 1$ large enough. ■

Proof of Proposition 5

We first give some notation. For every $\bar{a}$ in $[-1, 1]$, the best response $BR_{0,p}(\bar{a})$ is the solution of:

$$\max_a pu(a, 0) + (1-p)\, u(a, \bar{a}).$$

With the notation of the previous section, an element τ in T is such that τ_u is a Dirac measure on u. Then, τ is characterized by a distribution on A (that is τ_a). With a slight abuse of notation, we identify an element τ in T with its marginal τ_a on A.

We now prove the following statement:

Statement. Consider an interval of actions $[a_-, a_+]$ ($0 \in [a_-, a_+]$), an action that is a best response to some beliefs on $[a_-, a_+]$ putting at least probability p on 0 is an action in the interval $\left[a_-', a_+'\right]$ where

$$a_-' = \inf_{\bar{a} \in [a_-, a_+]} BR_{0,p}(\bar{a}) \text{ and } a_+' = \sup_{\bar{a} \in [a_-, a_+]} BR_{0,p}(\bar{a}),$$

Proof. The best response a of a player to belief on $[a_-, a_+]$ putting at least probability p on 0 solves a FOC

$$pu'_a(a,0) + (1-p)\int u'_a(a,\bar{a})\,dP(\bar{a}) = 0,$$

where dP is some Borel measure on $[a_-, a_+]$. Notice that the LHS of this FOC is an integral over the family of functions $pu'_a(a,0) + (1-p)u'_a(a,\bar{a})$ (indexed by $\bar{a}$). Furthermore, $BR_{0,p}(\bar{a})$ is characterized as the solution of $pu'_a(a,0) + (1-p)u'_a(a,\bar{a}) = 0$.The conclusion follows. ■

We are now in a position to define the sequence of sets $S^n_{0,p}$. To this purpose, denote $a^0_- = -1$ and $a^0_+ = +1$ and, for every $n \geq 1$, define iteratively the values a^n_- and a^n_+ in $[-1,1]$ by

$$\forall n \geq 1, a^n_- = \inf_{\bar{a}\in\left[a^{n-1}_-, a^{n-1}_+\right]} BR_{0,p}(\bar{a}) \text{ and } a^n_+ = \sup_{\bar{a}\in\left[a^{n-1}_-, a^{n-1}_+\right]} BR_{0,p}(\bar{a}),$$

(clearly, $0 \in \left[a^n_-, a^n_+\right]$ and $\left[a^n_-, a^n_+\right] \subset \left[a^{n-1}_-, a^{n-1}_+\right]$ for every n). Let T_p (that is $S^0_{0,p}$) denote is the set of distributions on $\left[a^0_-, a^0_+\right]$ putting at least probability p on 0. An action that is a best response to some beliefs in $S^0_{0,p}$ is an action in $\left[a^1_-, a^1_+\right]$ (from the Statement above). As every player is rational and has beliefs in $S^0_{0,p}$, the aggregate action is in $\left[a^1_-, a^1_+\right]$. Hence, $\phi\left(S^0_{0,p}\right)$ is the set of distributions on $\left[a^1_-, a^1_+\right]$. Define $S^1_{0,p} = \phi\left(S^0_{0,p}\right) \cap S^0_{0,p}$ is the set of distributions on $\left[a^1_-, a^1_+\right]$ putting at least probability p on 0.

A comment about this argument: the key point here is that p has 2 effects on the transition between $S^{n-1}_{0,p}$ and $S^n_{0,p}$: the "straight" effect that $S^n_{0,p}$ is a set of distributions on a subset X of actions putting at least probability p on one specific action (the equilibrium), and the other effect (on which the iterative contraction argument relies), that the support X on the distributions in $S^n_{0,p}$ shrinks with p (X decreases in p, for a given size of the support of $S^{n-1}_{0,p}$).

We now iterate the argument. If $S^{n-1}_{0,p}$ is the set of distributions on $\left[a^{n-1}_-, a^{n-1}_+\right]$ putting at least probability p on 0, then an action that is a best response to some beliefs in $S^{n-1}_{0,p}$ is an action in $\left[a^n_-, a^n_+\right]$ (from the Statement above). So $\phi\left(S^{n-1}_{0,p}\right)$ is then the set of distributions on $\left[a^n_-, a^n_+\right]$ and $S^n_{0,p} = \phi\left(S^{n-1}_{0,p}\right) \cap S^0_{0,p}$ is the set of distributions on $\left[a^n_-, a^n_+\right]$ putting at least probability p on 0.

By a standard argument, the two sequences a^n_- and a^n_+ converge, and $S^\infty_{0,p}$ is the set of distributions on $\left[a^\infty_-, a^\infty_+\right]$ putting at least probability p on 0. $S^\infty_{0,p}$ reduces to the equilibrium iff $a^\infty_- = a^\infty_+ = 0$. If $S^\infty_{0,p}$ does not reduce to the equilibrium, then every distribution on $\left[a^\infty_-, a^\infty_+\right]$ putting at least probability p on 0 is a $p-$consensus distribution.

A necessary condition for $a^\infty_- = a^\infty_+ = 0$ is that $BR_{0,p}$ is locally contracting at 0, that is:

$$\left|BR'_{0,p}(0)\right| < 1.$$

By the implicit functions theorem, we have:

$$BR'_{0,p}(\bar{a}) = -\frac{(1-p)\,u''_{a\bar{a}}\left(BR_{0,p}(\bar{a}),\bar{a}\right)}{pu''_{aa}\left(BR_{0,p}(\bar{a}),0\right) + (1-p)\,u''_{aa}\left(BR_{0,p}(\bar{a}),\bar{a}\right)}.$$

Then (given $BR_{0,p}(0) = 0$)

$$BR'_{0,p}(0) = -\frac{(1-p)\, u''_{a\bar{a}}(0,0)}{u''_{aa}(0,0)},$$

and the condition $\left|BR'_{0,p}(0)\right| < 1$ writes:

$$p > 1 - \left|\frac{u''_{aa}(0,0)}{u''_{a\bar{a}}(0,0)}\right|,$$

or, equivalently (differentiating the FOC $u'_a(BR(\bar{a}), \bar{a}) = 0$ at $(0,0)$):

$$p > 1 - \frac{1}{|BR'(0)|}. \tag{6}$$

On the other hand, a sufficient condition for $a_-^\infty = a_+^\infty = 0$ is that $BR_{0,p}$ is globally contracting:

$$\forall \bar{a} \in [-1,1], \left|BR'_{0,p}(\bar{a})\right| < 1.$$

If $M < 1$, then this condition holds true for every value of p and $\hat{p} = 0$. We assume $M \geq 1$ from now on. We have (given $u''_{aa} < 0$):

$$\left|BR'_{0,p}(\bar{a})\right| = \frac{(1-p)\left|u''_{a\bar{a}}(BR_{0,p}(\bar{a}), \bar{a})\right|}{p\left|u''_{aa}(BR_{0,p}(\bar{a}), 0)\right| + (1-p)\left|u''_{aa}(BR_{0,p}(\bar{a}), \bar{a})\right|},$$

and $\left|BR'_{0,p}(\bar{a})\right| < 1$ writes:

$$\frac{p}{1-p} > \left(\left|\frac{u''_{a\bar{a}}(BR_{0,p}(\bar{a}), \bar{a})}{u''_{aa}(BR_{0,p}(\bar{a}), \bar{a})}\right| - 1\right)\left|\frac{u''_{aa}(BR_{0,p}(\bar{a}), \bar{a})}{u''_{aa}(BR_{0,p}(\bar{a}), 0)}\right|.$$

The RHS of this inequality is smaller than $(M-1)\,m \geq 0$. It follows that the sufficient condition for convergence holds for every p such that $\frac{p}{1-p}$ is above this upper bound. Hence, the degree $\hat{p}$ of $p-$stability satisfies:

$$\frac{\hat{p}}{1-\hat{p}} < (M-1)\,m,$$

or, equivalently:

$$\hat{p} < 1 - \frac{1}{1+(M-1)\,m}. \tag{7}$$

The existence of $\hat{p}$ is shown in Proposition 2. Inequalities (6) and (7) imply the first part of the proposition. Propositions 1 and 3 imply the result on $p-$consensus distributions. ■

Proof of Lemma 2.

As the economy is sequentially regular, using the implicit function theorem, we obtain the existence of a neighborhood $N(\hat{\pi}_2, \varepsilon)$ of $\hat{\pi}_2$ such that evaluated at the PFE $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$,

$$d\pi_2 = -\left(\int \partial_{\pi_2}\hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1, \hat{\mathbf{y}})di\right)^{-1} \int [\partial_y \hat{\mathbf{x}}_2^i d\hat{\mathbf{y}}^i + \partial_{x_1}\hat{\mathbf{x}}_2^i d\hat{\mathbf{x}}_1^i]di$$

where $d\hat{\mathbf{y}}^i = \partial_{\pi_1}\hat{\mathbf{y}}^i d\pi_1 + \partial_q \hat{\mathbf{y}}^i dq + \partial_{\pi_2}\hat{\mathbf{y}}^i d\mathbf{f}^i$ and $d\hat{\mathbf{x}}_1^i = \partial_{\pi_1}\hat{\mathbf{x}}_1^i d\pi_1 + \partial_q \hat{\mathbf{x}}_1^i dq + \partial_{\pi_2}\hat{\mathbf{x}}_1^i d\mathbf{f}^i$ for all $i \in I$ and $d\pi_1 = (\pi_1 - \hat{p}_1)$, $dq = (q - \hat{q})$ and $d\mathbf{f}^i = \left(\mathbf{f}^i - \hat{\pi}_2\right)$. As the PFE is sequentially regular, $J_{11}^{-1} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$ exists. From the market clearing conditions in the spot markets at $t = 1$, by computation, it is checked that $d\pi_1 = -z_1(\int \partial_{\pi_2}\hat{\mathbf{x}}_1^i d\mathbf{f}^i di) - z_2(\int \partial_{\pi_2}\hat{\mathbf{y}}^i d\mathbf{f}^i di)$ while $dq = -z_3(\int \partial_{\pi_2}\hat{\mathbf{x}}_1^i d\mathbf{f}^i di) - z_4(\int \partial_{\pi_2}\hat{\mathbf{y}}^i d\mathbf{f}^i di)$. It follows that setting

$$\mathbf{M}^i = \left(\int \partial_{\pi_2}\hat{\mathbf{x}}_2^i di\right)^{-1} \begin{bmatrix} (\int \left(\partial_y \mathbf{x}_2^i \partial_{p_1}\hat{\mathbf{y}}^i + \partial_{x_1}\hat{\mathbf{x}}_2^i \partial_{\pi_1}\hat{\mathbf{x}}_1^i\right) di)(z_1 \partial_{\pi_2}\hat{\mathbf{x}}_1^i + z_2 \partial_{\pi_2}\hat{\mathbf{y}}^i) \\ +(\int (\partial_y \mathbf{x}_2^i \partial_q \hat{\mathbf{y}}^i + \partial_{x_1}\hat{\mathbf{x}}_2^i \partial_q \hat{\mathbf{x}}_1^i) di)(z_3 \partial_{\pi_2}\hat{\mathbf{x}}_1^i + z_4 \partial_{\pi_2}\hat{\mathbf{y}}^i) \\ -(\partial_y \mathbf{x}_2^i \partial_{\pi_2}\hat{\mathbf{y}}^i + \partial_{x_1}\hat{\mathbf{x}}_2^i \partial_{\pi_2}\hat{\mathbf{x}}_1^i) \end{bmatrix}$$

for each $i \in I$ yields the desired conclusion.■

Proof of Proposition 8.

Let $B(\bar{\varepsilon}) = \{x \in \Re_{++}^{L_2 - 1} : \|x - \hat{\pi}_2\| < \bar{\varepsilon}\}$. By choosing $N(\hat{\pi}_2, \varepsilon) \subseteq B(\bar{\varepsilon})$ so that $\Pi_2^0 \subset B(\bar{\varepsilon})$, the first-order approximation used in part (i) applies to all assignments of expectations $\mathbf{f} : I \to \Pi_2^0$. It follows that

$$\Pi_{2,0}^n = \{\pi_2 \in N(\hat{\pi}_2, \varepsilon) : d\pi_2 = \int \mathbf{M}^i d\mathbf{f}^i di,\ for\ some\ \mathbf{f} : I \to \Pi_{2,0}^{n-1}\} \cap \Pi_{2,0}^{n-1}$$

for $n = 1, 2, 3, ...$ Let $\tilde{\mathbf{v}} : I \to \Pi_2^0$ be an assignment. For each $i \in I$, define $v'^i \in S(\bar{\varepsilon})$ by $sign\ v_l'^i = sign\ \tilde{\mathbf{v}}_l^i$. Suppose there exists $\bar{\varepsilon} > 0$ such that $\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\| < \bar{\varepsilon}$, for all assignment of expectations $\mathbf{v} : I \to S(\bar{\varepsilon})$. Observe that the map $\mathbf{v}' : I \to S(\bar{\varepsilon})$ is an assignment. As $\|.\|$ is a monotone vector norm, $\left\|\int \mathbf{M}^i d\tilde{\mathbf{v}}^i di\right\| \leq \left\|\int \mathbf{M}^i d\mathbf{v}'^i di\right\| < \bar{\varepsilon}$. Let $\gamma(1) = \sup_{v:\mathbf{I}\to\Pi_2^0} \frac{\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\|}{\bar{\varepsilon}}$. Then, $\gamma(1) < 1$ and $\Pi_{2,0}^1 \cap \Pi_{2,0}^0 \subseteq B(\gamma(1)\bar{\varepsilon})$. For $n = 1, 2, ...$ define $\gamma(n) = \sup_{v:\mathbf{I}\to\Pi_{2,0}^{n-1}} \frac{\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\|}{\bar{\varepsilon}}$. Observe that $\Pi_{2,0}^n \cap \Pi_{2,0}^{n-1} \subseteq B(\gamma(n)\bar{\varepsilon})$. Further, as $\|.\|$ is a monotone vector norm, $1 > \gamma(n-1) > \gamma(n)$. It follows that $\cap_{n\geq 0}\Pi_{2,0}^n \subseteq \cap_{n\geq 0} B(\gamma(n)\bar{\varepsilon}) \subseteq B(\gamma(0)^n \bar{\varepsilon}) = \{\hat{\pi}_2\}$. As $\{\hat{\pi}_2\} \subseteq \tilde{\Pi}_{2,0}$, it follows that $\tilde{\Pi}_{2,0} = \{\hat{\pi}_2\}$.

Next, suppose there exists $\bar{\varepsilon} > 0$ such that $\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\| > \varepsilon$, for all assignment of expectations $\mathbf{v} : I \to S(\varepsilon)$ where $\varepsilon \leq \bar{\varepsilon}$. Then, $\gamma(1) > 1$ and $B(\gamma(1)\bar{\varepsilon}) \subseteq \Pi_{2,0}^n \cap \Pi_{2,0}^{n-1}$ so that $B(\gamma(1)\bar{\varepsilon}) \subseteq \cap_{n\geq 0}\Pi_{2,0}^n$ so that $\tilde{\Pi}_{2,0} \neq \{\hat{\pi}_2\}$. Suppose there exists $\bar{\varepsilon} > 0$ such that $\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\| > \varepsilon$, for all assignment of expectations $\mathbf{v} : I \to S(\varepsilon)$, for each $\varepsilon \leq \bar{\varepsilon}$. For $\pi_2 \in N(\hat{\pi}_2, \bar{\varepsilon})$, there exists $\varepsilon' > 0$ and an assignment of expectations $\mathbf{v} : I \to N(\hat{\pi}_2, \varepsilon')$ such that $\int \mathbf{M}^i d\mathbf{v}^i di = \pi_2$. Denote the corresponding set $M_{\hat{\pi}_2,\mathbf{v}}$ and let $M_{\hat{\pi}_2} = \cup_{\mathbf{v}\in\mathbf{V}} M_{\hat{\pi}_2,\mathbf{v}}$. Note that $M_{\hat{\pi}_2} \subseteq N(\hat{\pi}_2, \varepsilon')$. Hence, for any π_2 in $N(\hat{\pi}_2, \varepsilon)$, whenever ε is small enough, $M_{\hat{\pi}_2} \subseteq N(p_2, \varepsilon')$, so that $M_{\hat{\pi}_2} = M_{\pi_2}$; moreover, $M_{\pi_2} \subseteq \tilde{\Pi}_{2,0}$ for each $\pi_2 \in N(\hat{\pi}_2, \varepsilon'')$, and hence, $N(\hat{\pi}_2, \varepsilon'') \subseteq \tilde{\Pi}_{2,0}$ for all $\varepsilon'' \leq \bar{\varepsilon}$. By assumption, $\tilde{\Pi}_{2,0} \subseteq N(\hat{\pi}_2, \bar{\varepsilon})$, by continuity in p, there exists $\tilde{p} > 0$ such that for all $p < \tilde{p}$, $\tilde{\Pi}_{2,p} \subseteq N(\hat{\pi}_2, \bar{\varepsilon})$.

To check that the condition that $\left\|\int \mathbf{M}^i d\mathbf{v}^i di\right\| < \bar{\varepsilon}$ is invariant to the choice of the second period numeraire, note that multiplying all prices by the same positive scalar $\beta > 0$ implies that $\bar{\varepsilon}$ on the right hand side of the condition is now $\beta\bar{\varepsilon}$ while the expression on the left-hand side is equal to

$$\left\| \int -\beta \left(\int \partial_{\pi_2} \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1, \hat{\mathbf{y}}) di \right)^{-1} [\partial_y \hat{\mathbf{x}}_2^i d\hat{\mathbf{y}}^i + \partial_{x_1} \hat{\mathbf{x}}_2^i d\hat{\mathbf{x}}_1^i] di \right\| = \beta \left\| \int M^i \mathbf{v}^i di \right\|$$

for all $\mathbf{v} : I \to S(\bar{\varepsilon})$ which implies that the condition $\left\| \int \mathbf{M}^i d\mathbf{v}^i di \right\| < \bar{\varepsilon}$ itself remains unchanged.

Next, for a fixed assignment of expectations $\mathbf{v} : I \to \Pi_2^0$, $\Pi_2^0 = N(\hat{\pi}_2, \varepsilon)$, for some sequentially regular PFE $(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)$, we can write

$$\begin{aligned} \left\| \int \mathbf{M}^i \mathbf{v}^i di \right\| &= \left\| - \left(\int \partial_{\pi_2} \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1, \hat{\mathbf{y}}) \right)^{-1} \int [\partial_y \hat{\mathbf{x}}_2^i d\hat{\mathbf{y}}^i + \partial_{x_1} \hat{\mathbf{x}}_2^i d\hat{\mathbf{x}}_1^i] di \right\| \\ &= \left\| - \left(\int \partial_{\pi_2} \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1, \hat{\mathbf{y}}) \right)^{-1} \right\| \left\| \int [\partial_y \hat{\mathbf{x}}_2^i d\hat{\mathbf{y}}^i + \partial_{x_1} \hat{\mathbf{x}}_2^i d\hat{\mathbf{x}}_1^i] di \right\|. \end{aligned}$$

As market clearing in both periods is common knowledge, $\int \hat{\mathbf{y}}^i di = 0$ and $\int \hat{\mathbf{x}}_1^i di = \bar{\mathbf{w}}_1$ and therefore, $\int d\hat{\mathbf{y}}^i di = \int d\hat{\mathbf{x}}_1^i di = 0$. If both

$$\partial_y \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1^i, \hat{\mathbf{y}}^i(\hat{\pi}_1, \hat{q}, \hat{\pi}_2))) = \partial_y \hat{\mathbf{x}}_2^j(\hat{\mathbf{x}}_1^i, \hat{\mathbf{y}}^i(\hat{\pi}_1, \hat{q}, \hat{\pi}_2))) = \partial_y \hat{\mathbf{x}}_2(\hat{\mathbf{x}}_1^i, \hat{\mathbf{y}}^i(\hat{\pi}_1, \hat{q}, \hat{\pi}_2)))$$

and

$$\partial_{x_1} \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1^i, \hat{\pi}_2, \hat{\mathbf{y}}^i) = \partial_{x_1} \hat{\mathbf{x}}_2^j(\hat{\mathbf{x}}_1^i, \hat{\pi}_2, \hat{\mathbf{y}}^i) = \partial_{x_1} \hat{\mathbf{x}}_2(\hat{\mathbf{x}}_1^i, \hat{\pi}_2, \hat{\mathbf{y}}^i)$$

for all $i, j \in I$, $\int [\partial_y \hat{\mathbf{x}}_2^i d\hat{\mathbf{y}}^i + \partial_{x_1} \hat{\mathbf{x}}_2^i d\hat{\mathbf{x}}_1^i] di = \partial_y \hat{\mathbf{x}}_2 \int d\hat{\mathbf{y}}^i di + \partial_{x_1} \hat{\mathbf{x}}_2 \int d\hat{\mathbf{x}}_1^i di = 0$, which, in turn, implies that $\left\| \int M^i \mathbf{v}^i di \right\| = 0$ for every $\mathbf{v} : I \to S(\varepsilon)$, and every $\varepsilon > 0$ such that $S(\varepsilon) \subset \Pi_2^0$. Therefore, for $n = 1, 2, ...$, $\Pi_2^n = \{\hat{\pi}_2\}$. By continuity of $\|.\|$, there is an $\tilde{\varepsilon}_1 > 0$ such that if

$$\max \left\{ \left\| \partial_y \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1^i, \hat{\pi}_2, \hat{\mathbf{y}}^i) - \partial_y \hat{\mathbf{x}}_2^j(\hat{\mathbf{x}}_1^i, \hat{\pi}_2, \hat{\mathbf{y}}^i) \right\|, \left\| \partial_{x_1} \hat{\mathbf{x}}_2^i(\hat{\mathbf{x}}_1^i, \hat{\pi}_2, \hat{\mathbf{y}}^i) - \partial_{x_1} \hat{\mathbf{x}}_2^j(\hat{\mathbf{x}}_1^i, \hat{\pi}_2, \hat{\mathbf{y}}^i) \right\| \right\} < \tilde{\varepsilon}_1$$

for all $i, j \in I$, there exists $\varepsilon > 0$ such that $S(\varepsilon) \subset \Pi_2^0$ and $\left\| \int M^i \mathbf{v}^i di \right\| < \varepsilon$, *for all* $\mathbf{v} : I \to S(\varepsilon)$ and hence, $\tilde{\Pi}_{2,0} = \tilde{\Pi}_{2,p} \{\hat{\pi}_2\}$, $0 \le p \le 1$. By lemma 2:

$$\mathbf{M}^i = \left(\int \partial_{\pi_2} \hat{\mathbf{x}}_2^i di \right)^{-1} \begin{bmatrix} (\int \left(\partial_y \mathbf{x}_2^i \partial_{p_1} \hat{\mathbf{y}}^i + \partial_{x_1} \hat{\mathbf{x}}_2^i \partial_{\pi_1} \hat{\mathbf{x}}_1^i \right) di)(z_1 \partial_{\pi_2} \hat{\mathbf{x}}_1^i + z_2 \partial_{\pi_2} \hat{\mathbf{y}}^i) \\ +(\int (\partial_y \mathbf{x}_2^i \partial_q \hat{\mathbf{y}}^i + \partial_{x_1} \hat{\mathbf{x}}_2^i \partial_q \hat{\mathbf{x}}_1^i) di)(z_3 \partial_{\pi_2} \hat{\mathbf{x}}_1^i + z_4 \partial_{\pi_2} \hat{\mathbf{y}}^i) \\ -(\partial_y \mathbf{x}_2^i \partial_{\pi_2} \hat{\mathbf{y}}^i + \partial_{x_1} \hat{\mathbf{x}}_2^i \partial_{\pi_2} \hat{\mathbf{x}}_1^i) \end{bmatrix}$$

Hence, by continuity of $\|.\|$, there is an $\tilde{\varepsilon}_2 > 0$ such that if $\max \left\{ \left\| \partial_{\pi_2} \hat{\mathbf{y}}^i \right\|, \left\| \partial_{\pi_2} \hat{\mathbf{x}}_1^i \right\| \right\} <$

$\tilde{\varepsilon}_2$, for all $i \in I$, there exists $\varepsilon > 0$ such that $S(\varepsilon) \subset \Pi_2^0$ and $\left\| \int M^i \mathbf{v}^i di \right\| < \varepsilon$, *for all* $\mathbf{v} : I \to S(\varepsilon)$. Finally, the set $\tilde{\varepsilon} = \min \{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2\}$. ■